CHAPTER

5

Quadratic Expressions

Quadratic Relations of the Form $y = ax^2 + bx + c$

- Expand and simplify second-degree polynomial expressions, using a variety of tools and strategies.
- Factor polynomial expressions involving common factors, trinomials, and differences of squares, using a variety of tools and strategies.

Vocabulary

perfect square trinomial
difference of squares
quadratic expression

Polynomial expressions are used to model many situations. For example, the expression $2x + 5$ could represent the length of a basketball court or the number of seats in a theatre. Then, the expression $2x^2 + 11x + 15$ might represent the area of the basketball court or the revenue from theatre ticket sales.

In this chapter, you will learn how to multiply and factor polynomial expressions. In particular, you will develop skills for working with second-degree polynomials, or quadratic expressions.

Chapter Problem

A pedestal for trophies is to be made from three layers, each in the shape of a square-based prism with the same height. The pedestal can be made in any size, depending on the height, x, in centimetres.

How can you find the surface area of the pedestal?

Get Ready

Classify Polynomials

You can classify a **polynomial** by its number of terms or its degree.

The **degree of a polynomial** is the greatest degree of any of its terms.
The degree of a term is the sum of the exponents on its variables.

$2abc$ is a **monomial**, because it has one term.
The sum of the exponents is $1 + 1 + 1$, or 3. $2abc$ is a third-degree polynomial.

$7x^2 + x$ is a **binomial**, because it has two terms.
The greatest power is 2 from the term $7x^2$. $7x^2 + x$ is a second-degree polynomial.

$7k^2m + 15k^3m^2 - 6km^2$ is a **trinomial**, because it has three terms.
The greatest exponent sum is $3 + 2$, or 5, from the term $15k^3m^2$.
$7k^2m + 15k^3m^2 - 6km^2$ is a fifth-degree polynomial.

1. Classify each polynomial by its number of terms.

a) $-3y$

b) $5 + 6a^3$

c) $6x^2 + x - 1$

d) $8a^4b^4 - 6a^3b^2 + 2ab^2$

e) $5d^3e - 7e$

f) $19m + 8n - 3p$

2. State the degree of each polynomial.

a) $9 + 5y^5 - 4y^2 + y$

b) $8a^3b^2 + 9a^2b - 6a^4b^2$

c) $10x^7y^2 - 3x^3y^3 + 5x^4y^4$

d) $6abc - 5a^2bc^2 - 7abc^2$

Add and Subtract Polynomials

To add polynomials, remove the brackets and then collect **like terms**.

$$\begin{aligned}&(2x^2 + 3x - 5) + (7x^2 - 6x - 2)\\ &= 2x^2 + 3x - 5 + 7x^2 - 6x - 2\\ &= 2x^2 + 7x^2 + 3x - 6x - 5 - 2\\ &= 9x^2 - 3x - 7\end{aligned}$$

To subtract polynomials, add the opposite polynomial.

$$\begin{aligned}&(4a^2 + 5ab - 9b^2) - (7a^2 - 6ab + 2b^2)\\ &= (4a^2 + 5ab - 9b^2) + (-7a^2 + 6ab - 2b^2)\\ &= 4a^2 + 5ab - 9b^2 - 7a^2 + 6ab - 2b^2\\ &= 4a^2 - 7a^2 + 5ab + 6ab - 9b^2 - 2b^2\\ &= -3a^2 + 11ab - 11b^2\end{aligned}$$

3. Simplify.

a) $(5x + 7) + (2x - 11)$

b) $(3b - 8) - (6b - 7)$

c) $(5x^2 + 6x + 8) + (2x^2 + 5x - 9)$

d) $(9y^3 - 7y^2 + 6) - (3y^3 - 5y^2 + 8)$

e) $(7a^2 + 3a - 4) + (8a^2 - 2a - 15)$

f) $(2c^2 - 3c + 1) - (-c^2 - 3c - 5)$

4. Simplify.

a) $(7x^2 + 3xy - 2y^2) + (8x^2 - xy - y^2)$

b) $(4g^2 + gh - 7h^2) - (g^2 - 2gh + 3h^2)$

c) $(5ab^2 + 7a - b) + (3ab^2 - 5a + 6b)$

d) $(3cd^2 + 2c + 9d) - (2cd^2 + 2c - d)$

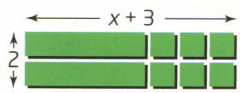
e) $(2x + 8) - (6x - 7) + (5x - 1)$

f) $(5a^2 - b) + (6b - 2a^2) - (b^2 + 7a^2)$

The Product of a Monomial and a Polynomial

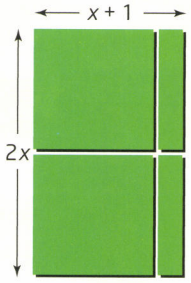
The **distributive property** allows you to expand algebraic expressions. When distributing, multiply the monomial by each term in the polynomial.

$$2(x + 3)$$
$$= 2(x) + 2(3)$$
$$= 2x + 6$$



The area is $2x + 6$.

$$2x(x + 1)$$
$$= 2x(x) + 2x(1)$$
$$= 2x^2 + 2x$$

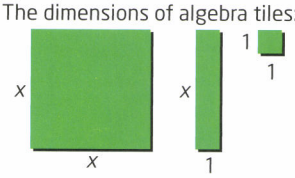


The area is $2x^2 + 2x$.

$$-a(3a + 5)$$
$$= -a(3a) + (-a)(5)$$
$$= -3a^2 + (-5a)$$
$$= -3a^2 - 5a$$

5. Use algebra tiles to model each product.

The dimensions of algebra tiles:



a) $3(x + 2)$

b) $4(x + 2)$

c) $x(x + 3)$

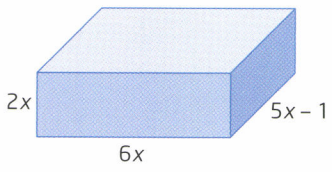
d) $4x(x + 4)$

6. Expand using the distributive property.

a) $7m(3m + 8)$ **b)** $-4(c + 9)$

c) $5a^2(6a^2 - 8a)$ **d)** $2(d^2 - 2d + 1)$

7. A rectangular prism has the dimensions shown.



a) Find a simplified expression for the volume.

b) Find a simplified expression for the surface area.

Factors

The **factors** of 12 are 1, 2, 3, 4, 6, and 12. To find the **greatest common factor** (GCF) of 12 and 18, express each number as the product of its prime factors.

$12 = (2) \times 2 \times (3)$

$18 = (2) \times (3) \times 3$

Since both 12 and 18 have factors of 2 and 3, their GCF is 2×3, or 6.

8. Write the factors of each number.

a) 10 **b)** 24 **c)** 16 **d)** 32

9. Write each number as the product of its prime factors.

a) 8 **b)** 14 **c)** 28 **d)** 30

10. Find the GCF of each pair of numbers.

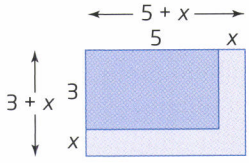
a) 6 and 9 **b)** 25 and 15

c) 24 and 16 **d)** 20 and 28

e) 36 and 15 **f)** 32 and 40

5.1 Multiply Polynomials

A rectangular garden measures 3 m by 5 m. If each dimension is increased by the same amount to expand the garden, how can you model the area of the new garden using a polynomial?



Investigate A

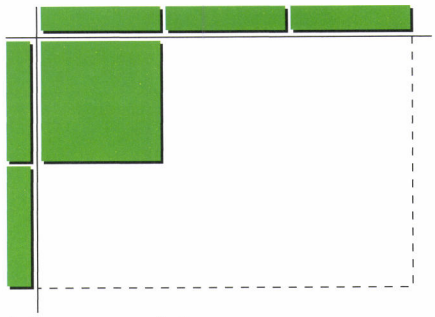
How can you model the multiplication of polynomials?

Tools

- algebra tiles

Method 1: Use Algebra Tiles

1. To show the product $(2x)(3x)$, use algebra tiles to model $2x$ and $3x$ as the dimensions of a rectangle. Then, fill in the rectangle with tiles to find an expression for the area. An x^2-tile has been placed to begin the process. How many x^2-tiles are needed to fill in the rectangle? What is the area of the rectangle? What is the product $(2x)(3x)$?



2. To show the product $(2x)(3x + 3)$, use algebra tiles to model $2x$ and $3x + 3$ as the dimensions of a rectangle.

a) Use two x-tiles to form one dimension along the left side. Use three x-tiles and three unit tiles to form the other dimension along the top. Complete the area of the rectangle.

b) How many x^2-tiles and x-tiles are needed? What is the resulting product?

3. Use algebra tiles to model the product $(3x)(2x + 1)$. What is the resulting product?

4. Use the same process to model the product of two binomials. What is each resulting product?

   a) $(x + 1)(x + 2)$ **b)** $(x + 2)(x + 4)$

   c) $(x + 3)(2x + 1)$ **d)** $(2x + 3)(x + 1)$

5. **Reflect** Consider your results from step 4.

   a) Describe how you can use algebra tiles to multiply two binomials.

   b) How are the terms in the resulting products related to the terms of the two binomials? Write a general rule for multiplying two binomials.

6. Use your rule to find each product.

   a) $(x + 3)(x + 8)$ **b)** $(2x + 5)(x + 4)$

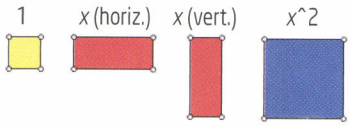
   c) $(4x + 7)(3x + 1)$

Method 2: Use *The Geometer's Sketchpad*®

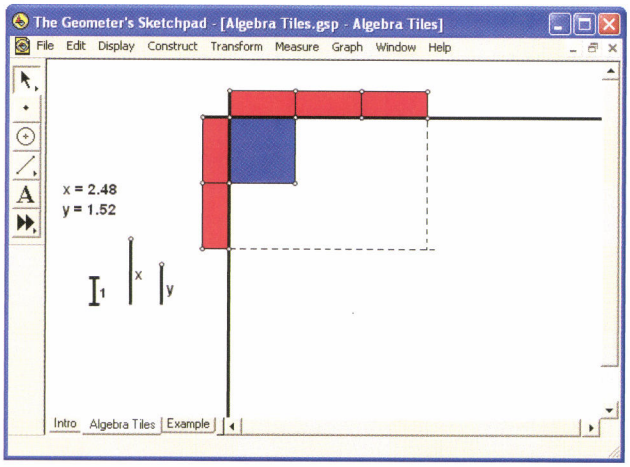
Tools

- computer with *The Geometer's Sketchpad*®
- Algebra Tiles.gsp

For this activity, you will use unit tiles, x-tiles (horizontal and vertical), and x^2-tiles.



1. To show the product $(2x)(3x)$, use virtual algebra tiles to model $2x$ and $3x$ as the dimensions of a rectangle. Then, fill in the rectangle with tiles to find an expression for the area.
   - Click and hold the **Custom Tool** icon. Select **x (vertical)** and place two tiles along the vertical line, as shown.
   - Repeat for **x (horizontal)**, placing three tiles along the horizontal line, as shown.
   - Repeat for **x^2** (x^2-tile) to complete the area of the rectangle.



How many x^2-tiles are needed? What is the area of the rectangle? What is the product $(2x)(3x)$?

2. Model the product $2x(3x + 3)$ using virtual algebra tiles.
   a) Place two **x (vertical)** tiles along the vertical line. Place three **x (horizontal)** tiles and three unit tiles along the horizontal line.
   b) Complete the area of the rectangle.
   c) How many x^2-tiles and x-tiles are needed? What is the resulting product?

3. Use virtual algebra tiles to illustrate the product $3x(2x + 1)$. What is the resulting product?

4. Use the same process to model the product of two binomials. What is each resulting product?
   a) $(x + 1)(x + 2)$ b) $(x + 2)(x + 4)$
   c) $(x + 3)(2x + 1)$ d) $(2x + 3)(x + 1)$

5. **Reflect** Consider your results from step 4.
   a) Describe how you can use virtual algebra tiles to multiply two binomials.
   b) How are the terms in the resulting products related to the terms of the two binomials? Write a general rule for multiplying two binomials.

6. Use your rule to find each binomial product.
   a) $(x + 3)(x + 8)$ b) $(2x + 5)(x + 4)$
   c) $(4x + 7)(3x + 1)$

Investigate B

Tools
- TI-89 calculator

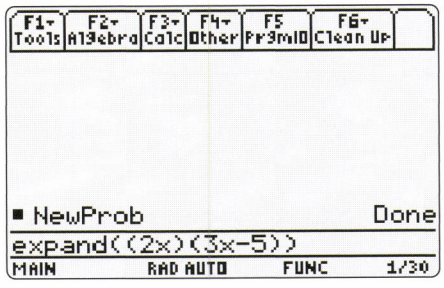
How can you relate the distributive property to polynomial multiplication?

Turn on the TI-89 calculator. If necessary, press the HOME key to display the computer algebra system (CAS) Home screen.

1. Clear the calculator's memory.
   - Press 2nd [F6] to display the **Clean Up** menu.
   - Select **2:NewProb**.
   - Press ENTER.

2. Expand $2x(3x - 5)$.
   - Press F2.
   - Select **3:expand(**.
   - Type $(2x)(3x - 5)$.
   - Press) ENTER.

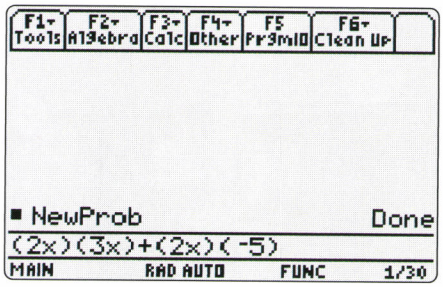
   What expression results?



3. Apply the distributive property. Multiply $2x$ by each term in the binomial $(3x - 5)$.

- Type $(2x)(3x) + (2x)(-5)$.
- Press ENTER.

What expression results?



4. Compare your answers for steps 2 and 3. Describe the process the calculator used to expand the expression $2x(3x - 5)$.

5. Use a CAS to expand each expression.

a) $2x(5x^2 - 3x + 1)$

b) $-3x^2(2x^2 + 5x - 3)$

c) $3xy(2x^2y - 4xy^2)$

d) $ax(bx - cy + d)$

Technology Tip

To enter exponents, use the ^ key. For example, to enter x^2, press X ^ 2.

6. Use a CAS to apply the distributive property to expand each expression in step 5. Verify that the answers are the same.

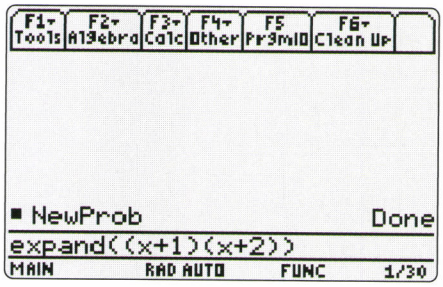
7. Use a CAS to expand the product of two binomials. What is each resulting product?

a) $(x + 1)(x + 2)$

b) $(x + 2)(x + 4)$

c) $(x + 3)(2x + 1)$

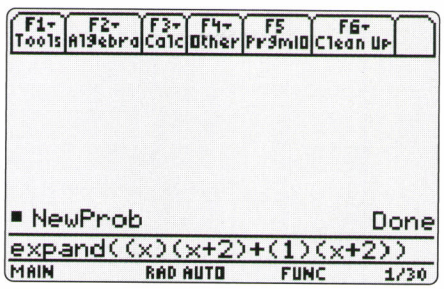
d) $(2x + 3)(x + 1)$



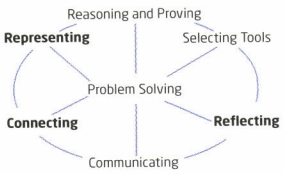
Technology Tip

To enter an expression containing variables multiplied together, you must use the × key. For example, to enter xy, press X × Y.

8. Use a CAS to apply the distributive property to expand each expression in step 7. Multiply each term of the first binomial by the second binomial. Verify that the answers are the same.



9. Reflect Consider your results from steps 7 and 8. How are the terms in the resulting products related to the terms of the two binomials? Write a general rule for multiplying two binomials.



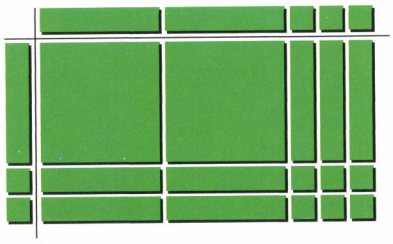
Example 1 Model a Binomial Product

Model the binomial product $(x + 2)(2x + 3)$.

Solution

Method 1: Use Algebra Tiles

Use algebra tiles to create a rectangle with width $x + 2$ and length $2x + 3$.



There are two x^2-tiles, seven x-tiles, and six unit tiles.

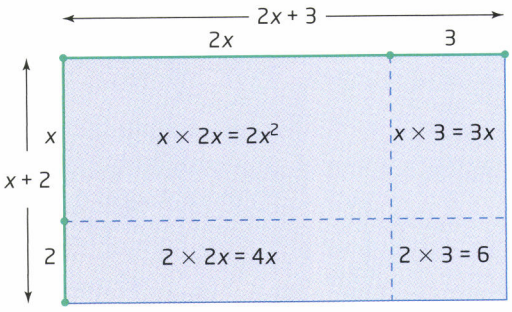
The area of the rectangle is $2x^2 + 7x + 6$.
$(x + 2)(2x + 3) = 2x^2 + 7x + 6$

Method 2: Use a Diagram

Draw a vertical segment of any length and label it $x + 2$ units. Divide the segment into a section x units long and a section 2 units long.

Perpendicular to the top of the vertical segment, draw a horizontal segment, $2x + 3$ units long, divided into a section $2x$ units long and a section 3 units long.

Complete the rectangle by drawing sides opposite $x + 2$ and $2x + 3$. Then, draw horizontal and vertical dashed segments from the section marks to the opposite sides, as shown. Find the areas of the four sections that make up the whole rectangle.



The binomial product $(x + 2)(2x + 3)$ equals the sum of the four areas.
$(x + 2)(2x + 3) = 2x^2 + 3x + 4x + 6$
$= 2x^2 + 7x + 6$

Example 2 Use the Distributive Property

Find each binomial product.

a) $(x + 2)(x + 5)$

b) $(x - 2)(x + 4)$

c) $(3x + 7)(x - 5)$

Solution

a) To multiply two binomials, use the distributive property. Then, simplify by collecting like terms.

$$\begin{aligned}&(x + 2)(x + 5)\\ &= x(x + 5) + 2(x + 5)\\ &= x(x) + x(5) + 2(x) + 2(5)\\ &= x^2 + 5x + 2x + 10\\ &= x^2 + 7x + 10\end{aligned}$$

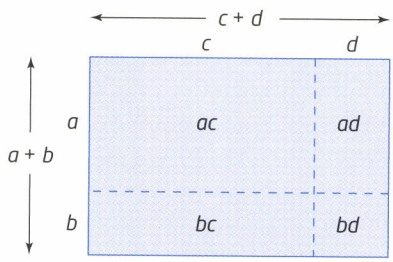
b)
$$\begin{aligned}&(x - 2)(x + 4)\\ &= x(x + 4) - 2(x + 4)\\ &= x^2 + 4x - 2x - 8\\ &= x^2 + 2x - 8\end{aligned}$$

c) Apply the distributive property mentally. Perform the multiplications indicated by the upper arrows and then the lower arrows. Then, simplify by collecting like terms.

$$\begin{aligned}&(3x + 7)(x - 5)\\ &= 3x^2 - 15x + 7x - 35\\ &= 3x^2 - 8x - 35\end{aligned}$$

You can find the product of two binomials by multiplying each term in the first binomial by each term in the second binomial. If necessary, simplify by collecting like terms.

$$(a + b)(c + d) = ac + ad + bc + bd$$



Example 3 Expand and Simplify

Expand and simplify.

a) $-2(4x - 5)(7x - 6)$

b) $2(x + 7)(x - 3) - (4x + 3)(2x - 1)$

Solution

a) First, find the product of the binomials. Then, multiply by -2.

$$-2(4x - 5)(7x - 6)$$
$$= -2(28x^2 - 24x - 35x + 30)$$
$$= -2(28x^2 - 59x + 30)$$
$$= -56x^2 + 118x - 60$$

b) $2(x + 7)(x - 3) - (4x + 3)(2x - 1)$
$= 2(x^2 - 3x + 7x - 21) - (8x^2 - 4x + 6x - 3)$
$= 2(x^2 + 4x - 21) - (8x^2 + 2x - 3)$
$= 2(x^2 + 4x - 21) - 1(8x^2 + 2x - 3)$
$= 2x^2 + 8x - 42 - 8x^2 - 2x + 3$
$= -6x^2 + 6x - 39$

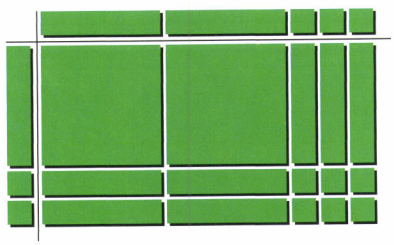
Multiplying a polynomial by -1 produces the opposite polynomial:
$(-1)(8x^2 + 2x - 3)$
$= -8x^2 - 2x + 3$

Literacy Connections

Expand, find the product, and multiply all mean the same thing.

Key Concepts

- You can model a binomial product as the area of a rectangle with the binomials as the dimensions.



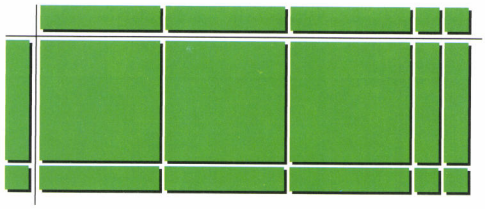
The area of the rectangle is $2x^2 + 7x + 6$.
$(x + 2)(2x + 3) = 2x^2 + 7x + 6$

- You can find the product of two binomials by multiplying each term in one binomial by each term in the other binomial. If necessary, simplify by collecting like terms.

$$(x + 2)(2x + 3)$$
$$= 2x^2 + 3x + 4x + 6$$
$$= 2x^2 + 7x + 6$$

Communicate Your Understanding

C1 Explain how the algebra tiles illustrate the product $(x + 1)(3x + 2)$.



C2 Jason expands and simplifies $(x + 3)(x + 7)$ in his head. He adds 3 and 7 to get $10x$, and multiplies 3 and 7 to get 21. His final answer is $x^2 + 10x + 21$.

a) Explain Jason's logic.

b) Would Jason's method work when expanding $(2x + 3)(5x + 6)$? If yes, explain why. If no, how should he adapt his method?

c) Would this method work for negative constant terms? Explain.

C3 Describe the steps you would use to expand $(3x + 5)(2x - 9)$.

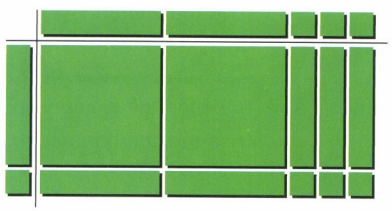
C4 Rolly uses an acronym to help him remember how to expand two binomials. It is called the FOIL method, for *First*, *Outside*, *Inside*, *Last*. What does this mean?

Practise

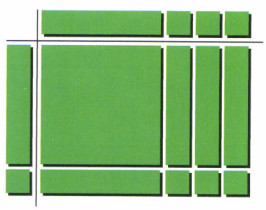
For help with questions 1 and 2, see Example 1.

1. What binomial product does each model illustrate?

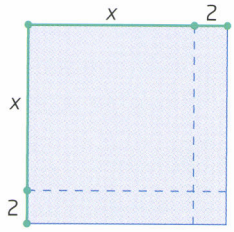
a)



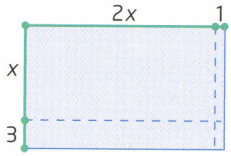
b)



c)



d)



2. Model each binomial product using algebra tiles, virtual algebra tiles, or a diagram.

a) $(2x + 1)(x + 1)$ **b)** $(x + 4)(x + 2)$

c) $(x + 1)(x + 5)$ **d)** $(2x + 1)(3x + 2)$

For help with questions 3 to 6, see Example 2.

3. Use the distributive property to find each binomial product.

a) $(x + 3)(x + 5)$ **b)** $(x + 3)(x + 4)$

c) $(y + 2)(y + 4)$ **d)** $(r + 4)(r + 2)$

e) $(n + 7)(n + 1)$ **f)** $(p + 9)(p + 9)$

g) $(w + 7)(w + 8)$ **h)** $(d + 3)(d + 11)$

4. Use the distributive property to find each binomial product.

a) $(k - 3)(k - 5)$ **b)** $(y - 3)(y - 4)$

c) $(x - 2)(x - 4)$ **d)** $(q - 4)(q - 2)$

e) $(j - 7)(j - 1)$ **f)** $(p - 9)(p - 3)$

g) $(z - 7x)(z - 8x)$ **h)** $(b - 3c)(b - 11c)$

5. Use the distributive property to find each binomial product.

a) $(x + 3)(x - 5)$ **b)** $(y + 3)(y - 4)$

c) $(c - 2)(c + 4)$ **d)** $(w - 4)(w + 2)$

e) $(m + 7)(m - 1)$ **f)** $(y - 9)(y + 3)$

g) $(x + 7y)(x - 8y)$ **h)** $(a + 6b)(a - 10b)$

6. Find each binomial product.

a) $(2x + 3)(x + 4)$ **b)** $(y - 3)(5y - 7)$

c) $(6c - 1)(3c + 5)$ **d)** $(7w - 2)(2w + 1)$

e) $(5m + 6)(5m - 6)$

f) $(9y - 2)(2y + 2)$

g) $(7d + 5c)(8d - 6c)$

h) $(6q + 5r)(7q - 12r)$

For help with questions 7 and 8, see Example 3.

7. Expand and simplify.

a) $3(x - 5)(x + 6)$ **b)** $-2(x - 7)(x - 9)$

c) $-(y + 2)(y - 8)$ **d)** $2(k + 3)(k + 7)$

e) $m(m - 3n)(m - 5n)$

f) $2a(3a + 4b)(6a + 7b)$

8. Expand and simplify.

a) $(x + 4)(x + 6) + (x - 1)(x + 7)$

b) $(2x + 5)(3x - 7) + 2(4x + 9)(2x - 11)$

c) $3(6x - 2)(6x - 1) - (2x - 3)(5x + 6)$

d) $-(x - 2)(x - 3) + 2(3x + 5)(x + 4)$

e) $(x + 4)^2 - (x - 4)^2$

f) $-5(3x - 1)(5x - 2) + 6(6x + 3)(5x - 2)$

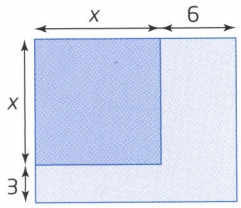
Connect and Apply

9. The predicted flight path of a firework is defined by the relation $h = -2(d - 3)(d - 15)$, where d is the horizontal distance, in metres, from a safety wall, and h is the height, in metres.

a) Expand and simplify the relation.

b) Verify that the relation from part a) is equivalent to the original relation. Use both relations to determine the height of the firework if d represents 10 m.

10. A square garden has side length x. One dimension is increased by 6 m and the other is increased by 3 m.



a) Write an algebraic expression for the area of the original garden.

b) Write an algebraic expression for the area of the new garden.

c) Expand and simplify your area expression from part b).

d) Find an expression that represents the increase in area.

e) If x represents 12 m, find the increase in area.

11. a) For each situation, begin with a square field, measuring x metres by x metres. Then, draw a diagram of the new field, write an algebraic expression for its area, and expand and simplify your area expression.

i) The length of one side is increased by 10 m.

ii) The length of one side is doubled.

iii) The length is increased by 5 m and the width is increased by 6 m.

b) **Use Technology** Use a CAS to verify your answers to part a).

12. A parabola has equation $y = (x + 3)(x - 1)$.

a) Find the x-intercepts of $y = (x + 3)(x - 1)$.

b) Expand and simplify the equation.

c) Graph the result from part b). Verify that the x-intercepts are the same.

Making Connections

In Chapter 4, you worked with quadratic relations of the form $y = (x - r)(x - s)$.

13. A rectangular prism has width w centimetres, length 2 cm more than its width, and height 2 cm.

a) Draw a diagram of the prism.

b) Express the volume as a product.

c) Expand and simplify the volume expression.

14. A cube has side length x. Each dimension is increased by y.

a) Draw a diagram of the cube.

b) Write an algebraic expression for the surface area of the original cube.

c) Write an algebraic expression for the surface area of the new cube.

d) Write an algebraic expression for the difference in surface area. Expand and simplify.

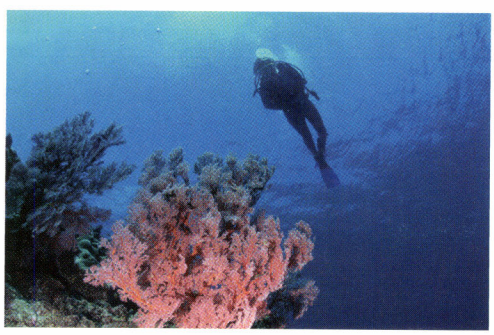
e) Write an algebraic expression for the difference in volume. Expand and simplify.

15. A scuba diver is drifting in a current of 0.3 m/s. If she swims with the current at an additional speed of v metres per second, the distance, d, in metres, that she travels before running out of air can be modelled by the relation $d = 3000(v + 0.3)(1.0 - v)$.

a) Expand this relation.

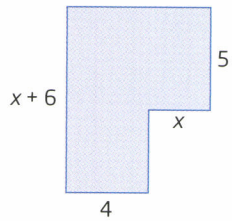
b) If she swims at 0.2 m/s, how far can she swim before running out of air?

c) **Use Technology** Use a graphing calculator or computer software to investigate what swimming speed would result in the maximum distance before running out of air.

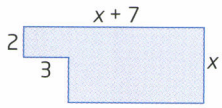


16. Write an algebraic expression for the area of each figure. Expand and simplify. Then, find the area in another way to verify your result.

a)



b)



Extend

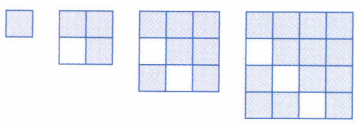
17. The number of hot dogs, n, sold by Wayne's Wiener World on a given day is modelled by $n = 500 - 100p$, where p is the price, in dollars.

a) Solve this equation for p.

b) The revenue generated by hot dog sales is $R = np$. Substitute your expression for p from part a), and expand to obtain an expression for the daily hot dog revenue.

c) **Use Technology** Use a CAS to verify your answer to part b).

18. Determine an algebraic expression for the number of shaded small squares in the nth diagram. Test your expression for two more diagrams.



19. Pose and solve a problem in which the dimensions of a rectangular object are written as binomials.

20. **Math Contest** When a positive integer n is divided by 24, the remainder is 18. When n is divided by 8 the remainder is

A 0 **B** 1 **C** 2 **D** 4 **E** 6

5.2 Special Products

The patterns in tile floors, brick patios, and quilts often repeat. Some quilts are made from square pattern blocks of material that are stitched together. The square pattern block shown starts with a square that is enlarged with the addition of rectangles to each of its dimensions until the desired square dimensions are reached. If the side length of the square pattern block is represented by $x + 3$, what is an expression for its area?

Investigate

How can you use patterns to find special products?

Tools

Optional

- algebra tiles

A: Squaring Binomials

Method 1: Use Pencil and Paper

1. Use algebra tiles or a diagram to square the binomial.
   $(x + 3)^2 = (x + 3)(x + 3)$

2. Expand and simplify.
   a) $(x + 3)^2$ **b)** $(x + 2)^2$ **c)** $(x - 6)^2$
   d) $(x - 4)^2$ **e)** $(2x + 5)^2$ **f)** $(3x - 1)^2$
   g) $(2x - 5y)^2$ **h)** $(4x + 7y)^2$

3. Consider each simplified expansion from step 2.
   a) How is the first term in each trinomial related to the first term in each binomial?
   b) How is the last term in each trinomial related to the last term in each binomial?
   c) How is the middle term in each trinomial related to the terms in the binomial?

4. **Reflect** Write a rule for expanding and simplifying $(a + b)^2$ or $(a - b)^2$.

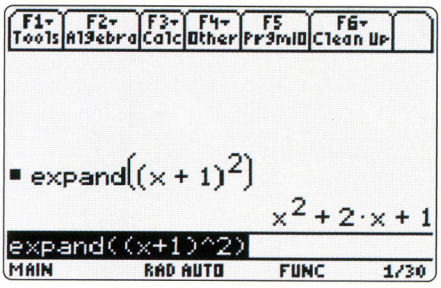
5. Use your rule to square each binomial.
   a) $(5x + 3y)^2$ **b)** $(7c - 4d)^2$

Method 2: Use a Computer Algebra System (CAS)

Tools
- TI-89 calculator

1. Clear the calculator's memory by selecting **2:NewProb** from the **Clean Up** menu.

2. Use the **Expand** function on each square of a binomial. Record the results.
   - **a)** $(x + 1)^2$
   - **b)** $(x + 2)^2$
   - **c)** $(x + 3)^2$
   - **d)** $(x + 4)^2$
   - **e)** $(x + 5)^2$



3. Compare the individual terms of the expansion to the binomial. Describe any patterns you notice, including any for the signs of the terms in the trinomial.

4. Use a CAS to expand each square of a binomial. Record the results.
   - **a)** $(2x + 2)^2$
   - **b)** $(2x + 3)^2$
   - **c)** $(2x - 4)^2$
   - **d)** $(2x - 5)^2$

5. Compare the individual terms of the expansion to the binomial. Describe any patterns you notice. How has your description of patterns changed compared to step 3?

6. Use a CAS to expand each square of a binomial. Record the results.
   - **a)** $(3a + 2)^2$
   - **b)** $(5m - 3)^2$
   - **c)** $(4 + 2b)^2$
   - **d)** $(7 - 3z)^2$
   - **e)** $(2x + 3y)^2$

7. Compare the individual terms of the expansion to the binomial. Have the patterns you described in step 5 changed?

8. **Reflect** Write a rule for expanding each square of a binomial.
   - **a)** $(a + b)^2$
   - **b)** $(a - b)^2$

B: Product of a Sum and a Difference of Two Terms

Method 1: Use Pencil and Paper

1. Expand and simplify.
   - **a)** $(x + 3)(x - 3)$
   - **b)** $(2y + 5)(2y - 5)$
   - **c)** $(x - 4)(x + 4)$
   - **d)** $(3k - 7)(3k + 7)$

2. How are the two binomials in each multiplication in step 1 alike? How are they different?

3. Consider each simplified expansion from step 1.
   a) How is the first term related to the first terms of the two binomials?
   b) How is the last term related to the last terms of the two binomials?
   c) Explain why there are only two terms in the simplified expansion.

4. **Reflect** Write a rule for expanding and simplifying $(a + b)(a - b)$. Does your rule apply to $(a - b)(a + b)$? Explain.

5. Use your rule to find the product of each sum and difference of two terms.
   a) $(2x + 3y)(2x - 3y)$
   b) $(5m + 7)(5m - 7)$

Tools
- TI-89 calculator

Method 2: Use a CAS

1. Clear the calculator's memory by selecting **2:NewProb** from the **Clean Up** menu.

2. Use the **Expand** function on each binomial product. Record the results.
   a) $(x + 2)(x - 2)$
   b) $(x - 3)(x + 3)$
   c) $(2x - 1)(2x + 1)$
   d) $(3x + 4)(3x - 4)$
   e) $(2x + 3y)(2x - 3y)$
   f) $(3m - 4n)(3m + 4n)$

3. How are the two binomials in each multiplication in step 2 alike? How are they different?

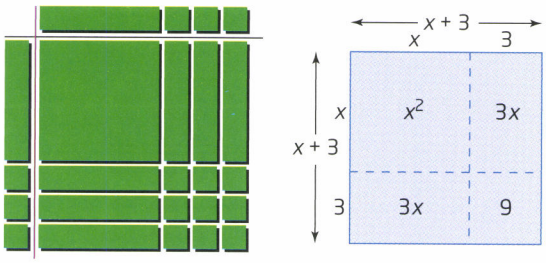
4. Consider each expansion from step 2.
   a) How is the first term related to the first terms of the two binomials?
   b) How is the last term related to the last terms of the two binomials?
   c) Explain why there are only two terms in the simplified expansion.

5. **Reflect** Write a rule for expanding and simplifying $(a + b)(a - b)$. Does your rule apply to $(a - b)(a + b)$? Explain.

While you can always use the distributive property to find the product of two binomials, some pairs of binomials result in special products that follow specific patterns.

Squaring Binomials

You can visualize squaring a binomial using algebra tiles or a diagram. Surrounding the x^2 region, there are *two* regions of $3x$, which combine to give $6x$, and a region defined by 3×3, or 9, giving $x^2 + 6x + 9$.



$$\begin{aligned}(x + 3)^2 &= (x + 3)(x + 3)\\ &= x^2 + 3x + 3x + 3^2\\ &= x^2 + 2(3x) + 9\\ &= x^2 + 6x + 9\end{aligned}$$

Alternatively, you can use the patterns for squaring a binomial. Square the first term, find twice the product of the terms, and square the last term. The middle term is added when you square a sum and subtracted when you square a difference.

$(a + b)^2 = a^2 + 2ab + b^2$ or $(a - b)^2 = a^2 - 2ab + b^2$

The resulting products are called **perfect square trinomials**.

perfect square trinomial

- a trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ that is the result of squaring a binomial

Product of a Sum and a Difference

When you multiply the sum and the difference of two terms, there are only two terms in the simplified expansion. There is no middle term because the other two products have a sum of zero.

$$\begin{aligned}(x + 3)(x - 3) &= x^2 - 3x + 3x - 3^2\\ &= x^2 - 9\end{aligned}$$

To find the product of the sum and the difference of two terms, use the pattern. Find the product of the first terms and the product of the second terms.

$(a + b)(a - b) = a^2 - b^2$

The resulting product, $a^2 - b^2$, is called a **difference of squares**.

I can change the order of the binomial factors and get the same result.
$(a - b)(a + b) = a^2 - b^2$
This happens because I can multiply two numbers or expressions in either order.

Literacy Connections

When you change the operation between the two terms of a binomial, the two forms are called conjugates. $x + 3$ and $x - 3$ are conjugates.

difference of squares

- an expression of the form $a^2 - b^2$ that involves the subtraction of two squares

Example 1 Apply Special Product Patterns

Expand and simplify.

a) $(x + 4)^2$ **b)** $(k - 5)^2$ **c)** $(3y + 7x)^2$

d) $(q - 11)(q + 11)$ **e)** $(4m + 3n)(4m - 3n)$

Solution

a) $(a + b)^2 = a^2 + 2ab + b^2$
$(x + 4)^2 = (x)^2 + 2(x)(4) + (4)^2$
$= x^2 + 8x + 16$

Use the appropriate pattern for squaring a binomial.

b) $(a - b)^2 = a^2 - 2ab + b^2$
$(k - 5)^2 = (k)^2 - 2(k)(5) + (5)^2$
$= k^2 - 10k + 25$

Use the appropriate pattern for squaring a binomial.

c) $(a + b)^2 = a^2 + 2ab + b^2$
$(3y + 7x)^2 = (3y)^2 + 2(3y)(7x) + (7x)^2$
$= 9y^2 + 42xy + 49x^2$

Use the appropriate pattern for squaring a binomial.

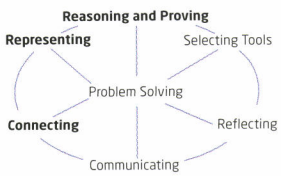
d) $(a - b)(a + b) = a^2 - b^2$
$(q - 11)(q + 11) = (q)^2 - (11)^2$
$= q^2 - 121$

Use the pattern for the product of a sum and a difference.

e) $(a + b)(a - b) = a^2 - b^2$
$(4m + 3n)(4m - 3n) = (4m)^2 - (3n)^2$
$= 16m^2 - 9n^2$

Use the pattern for the product of a sum and a difference.

Example 2 Helicopter Pad



The radius of a circular helicopter landing pad is increased by 3 m.

a) Find a simplified expression for the area of the new circle.

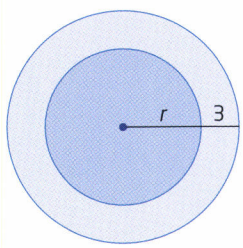
b) Find a simplified expression for the increase in area.

Solution

a) Area of the original circle $= \pi r^2$

Area of the larger circle $= \pi(r + 3)^2$
$= \pi[(r)^2 + 2(r)(3) + (3)^2]$
$= \pi(r^2 + 6r + 9)$
$= \pi r^2 + 6\pi r + 9\pi$

I can use the appropriate pattern for squaring a binomial.



b) Increase in area = (area of larger circle) − (area of original circle)
$= (\pi r^2 + 6\pi r + 9\pi) - (\pi r^2)$
$= 6\pi r + 9\pi$

Key Concepts

- When squaring a binomial, you add the two equal middle terms after expansion.
  $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$
- When you multiply the sum and the difference of two terms, the two middle terms are opposites, so they add to zero.
  $(a + b)(a - b) = a^2 - b^2$

Communicate Your Understanding

C1 Explain when the middle term is positive and when it is negative when you are squaring a binomial.

C2 Describe the steps you would use to expand and simplify each product.

a) $(x - 2)^2$ **b)** $(x - 2)(x + 2)$

Practise

1. Draw a diagram to illustrate each product.

a) $(x + 5)^2$ **b)** $(x + 6)^2$
c) $(x + a)^2$ **d)** $(ax + b)^2$

For help with questions 2 to 6, see Example 1.

2. Expand and simplify.

a) $(x + 5)^2$ **b)** $(y + 4)^2$ **c)** $(w + 6)^2$
d) $(k + 7)^2$ **e)** $(m + 11)^2$ **f)** $(c + 10)^2$
g) $(g + 9)^2$ **h)** $(x + 20)^2$

3. Expand and simplify.

a) $(x - 5)^2$ **b)** $(z - 3)^2$ **c)** $(x - 9)^2$
d) $(c - 1)^2$ **e)** $(v - 12)^2$ **f)** $(b - 100)^2$
g) $(n - 2)^2$ **h)** $(m - 6)^2$

4. Expand and simplify.

a) $(x + 3y)^2$ **b)** $(2x - y)^2$ **c)** $(5c + 2d)^2$
d) $(3a - 4b)^2$ **e)** $(9k + 2m)^2$ **f)** $(4u - 5v)^2$

5. Expand and simplify.

a) $(v + 1)(v - 1)$ **b)** $(a - 1)(a + 1)$
c) $(y + 5)(y - 5)$ **d)** $(x - 7)(x + 7)$
e) $(e - 9)(e + 9)$ **f)** $(z + 6)(z - 6)$
g) $(x + 12)(x - 12)$ **h)** $(y - 3)(y + 3)$

6. Expand and simplify.

a) $(w - v)(w + v)$ **b)** $(3m - n)(3m + n)$
c) $(y + 6x)(y - 6x)$ **d)** $(3x + 4y)(3x - 4y)$
e) $(7g - 3h)(7g + 3h)$
f) $(9x - 8y)(9x + 8y)$

Connect and Apply

7. Expand and simplify. Verify your answers using one of three methods:

- Check that substituting $x = 2$ into the original expression and the simplified expansion yields the same answer.
- Check that graphing both the original expression and the simplified expansion using a graphing calculator yields only one graph.
- Check that using a CAS to expand the original expression yields the same answer.

a) $(x + 4)(x - 4)$ **b)** $(x - 8)^2$
c) $(x + 8)^2$ **d)** $(x - 10)(x + 10)$
e) $(x + 11)(x - 11)$ **f)** $(x + 12)^2$
g) $(x - 7)^2$ **h)** $(x - 30)(x + 30)$

For help with questions 8 and 9, see Example 2.

8. The radius, r, of a circle has been increased by k. Both r and k are measured in the same units. Write a formula for the area of the new circle. Expand and simplify.

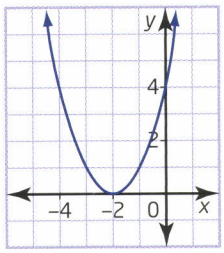
9. Each dimension of a square playground is increased by 5 m.

a) Draw a diagram of the situation.

b) Find a simplified algebraic expression for the area of the new playground.

c) Find a simplified algebraic expression for the increase in area.

10. A parabola has equation $y = (x + 2)^2$.



a) Identify the coordinates of the vertex.

b) Expand and simplify the equation.

c) Verify that the coordinates of the vertex satisfy the equation from part b).

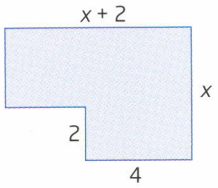
11. A square has side length $3x$. One dimension is increased by $2y$ and the other is decreased by $2y$.

a) Find an algebraic expression for the area of the resulting rectangle. Expand.

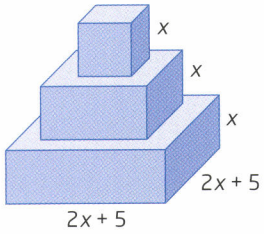
b) Find an algebraic expression for the change in area. Expand.

c) Calculate the area of the rectangle and change in area if x represents 8 cm and y represents 5 cm.

12. Use two methods to determine an algebraic expression to represent the area of the figure. Verify that they are equivalent expressions.



13. Chapter Problem A pedestal for trophies is to be made from three layers, each in the shape of a square-based prism. All three layers are the same height, x, in centimetres, but each base length is 3 cm less than that of the layer immediately below.



a) Write algebraic expressions for the side and top surface areas of each prism used to make the pedestal.

b) Write an algebraic expression for the exposed top surface area of the bottom layer of the pedestal. Expand and simplify.

c) Write an algebraic expression for the exposed top surface area of the middle layer of the pedestal. Expand and simplify.

14. An interesting way to multiply 19×21 is

$(20 - 1)(20 + 1)$
$= 20^2 - 1$
$= 400 - 1$
$= 399$

Use this method to calculate each product.

a) 31×29

b) 59×61

c) 99×101

d) 71×69

15. Explain how to adapt the method in question 14 to multiply 32×28. Then, use your method to multiply each product.

a) 76×84

b) 35×25

c) 104×96

d) 77×83

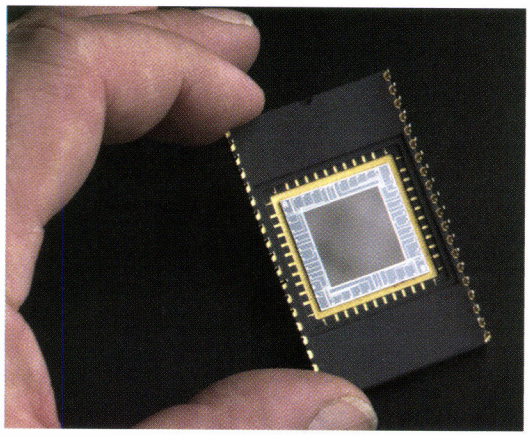
16. A stone is dropped from a height of 10 m. Its height as it falls can be approximated by the relation $h = -5t^2 + 10$, where t is the time, in seconds, and h is the height, in metres.

a) A delay of 3 s would cause the graph to shift 3 units to the right. Sketch a graph of this relation and of the relation after a 3-s delay.

b) Rewrite the relation to represent a delay of 3 s.

c) Expand the new relation and simplify.

17. Instead of film, a digital camera has an image sensor that converts light into electrical charges. The image sensor in most digital cameras is a charge coupled device (CCD). The CCD for an 8-megapixel digital camera measures 3264 pixels by 2448 pixels. A new model of CCD increases each dimension by x pixels.



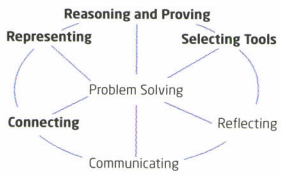
a) Write an expression for the number of pixels in the new CCD, and expand.

b) If x represents 1000 pixels, what is the resolution of the new CCD, in megapixels?

Literacy Connections

Mega is a prefix in the SI system of units for 10^6, or 1 000 000. For example, 1 megapixel equals one million pixels.

Achievement Check



18. a) Use algebra tiles or a diagram to expand $(a + b)^2$.

b) Use a diagram to expand $(a + b + c)^2$.

c) The length and width of a rectangle are represented by $x + 2$ and $9 - 4x$. If x must be an integer, what are the possible values for the area of the rectangle?

Extend

19. Expand and simplify. Use a CAS to verify your answers.

a) $(x - 2)^4$

b) $(2x + 3)(x - 5)(4x + 7)$

c) $(2x^2 + 5x + 3)^2$

d) $(5x - 2)^3$

20. The kinetic energy, E, in joules, of a moving object is given by the formula $E = \frac{1}{2}mv^2$, where m is the mass, in kilograms, of the object, and v is its speed, in metres per second.

a) Find an algebraic expression for the difference in kinetic energy of two objects with a difference in speed of 5 m/s.

b) Find an algebraic expression for the difference in kinetic energy of two objects with a difference in speed of x metres per second.

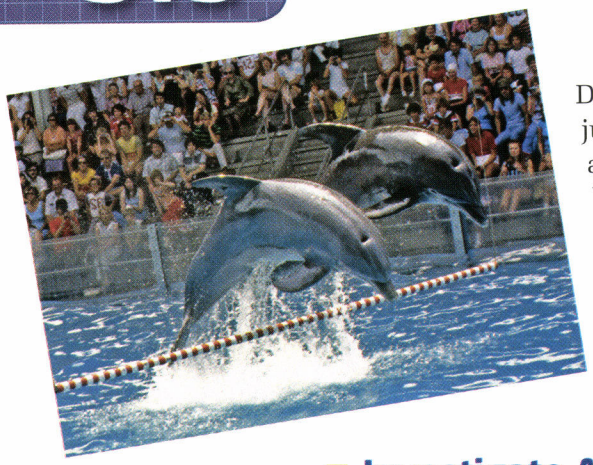
c) Expand each expression and simplify.

21. The sum of the cubes of the first n natural numbers can be found using the formula $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}$.

a) Verify that this is true for the first five natural numbers.

b) Show that the right side of the formula can also be expressed in the form $\left[\frac{n(n + 1)}{2}\right]^2$.

5.3 Common Factors



During a performance at a sea-life park, a dolphin jumps out of the water. Its height, h, in metres, above the water after t seconds can be approximated by the relation $h = 10x - 5x^2$. This relation can also be written as $h = 5x(2 - x)$, because the terms in the polynomial $10x - 5x^2$ have a common factor of $5x$.

Investigate A

Tools

- algebra tiles

How can you use a model to find factors of a polynomial?

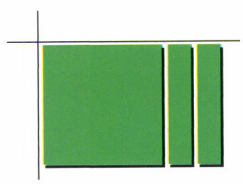
1. To factor $2x + 4$, use algebra tiles to create a rectangular area whose length and width represent the factors of the polynomial.

a) Arrange two x-tiles and four unit tiles to form a rectangle with area $2x + 4$. Then, place tiles along the left side and top to find the length and width of the rectangle. One dimension has been done for you.

b) Write an equation for the area as a product of the length and width.

2. Repeat step 1 for $6x + 18$. How many different rectangles can you find?

3. Use algebra tiles to find the factors of $x^2 + 2x$. Express the area as a product of the length and width.



4. Use algebra tiles to factor $2x^2 + 4x$. How many different rectangles can you find? Write an area statement for each one.

5. Use algebra tiles to factor each expression, if possible. If it is not possible, explain why.

a) $3x + 3$ **b)** $4x + 10$ **c)** $x^2 + 4x$

d) $2x^2 + 6x$ **e)** $2x + 5$ **f)** $4x^2 + 10x$

6. **Reflect** Explain how you can express a polynomial as a product of factors.

Investigate B

How can you use the greatest common factor (GCF) to factor a polynomial?

Method 1: Use Pencil and Paper

1. Find the GCF for each set of numbers by first expressing each number as a product of prime factors.

a) 12 and 8 **b)** 15 and 25

c) 4, 10, and 6 **d)** 6, 18, and 24

2. a) Find the GCF of 12 and 9.

b) Write each number as a product of two factors, where the first factor is the GCF. What operation did you use to obtain the second factor?

3. Find the GCF of each pair of terms.

a) 7^3 and 7^2 **b)** 5^6 and 5^4

c) x^2 and x **d)** x^3 and x^4

4. a) Find the GCF of x^6 and x^4.

b) Write each term as a product of two factors, where the first factor is the GCF. What operation did you use to obtain the second factor?

5. Find the GCF of the polynomial $2x^2 + 4x$ by first expressing each term as a product.

$2x^2 = 2 \times x \times x$

$4x = 2 \times 2 \times x$

Multiply the common factors to calculate the GCF.

a) What is the GCF of $2x^2 + 4x$?

b) Rewrite the polynomial as the sum of products. Express each term as a product of two factors, where the first factor is the GCF. Use division to determine the second factor of each term.

c) Write the polynomial as a product of two factors, where the first factor is the GCF. What polynomial is the second factor?

d) Verify your factors from part c) by expanding.

6. Repeat step 5 for each polynomial.

a) $3x^2 + 6x$ **b)** $2x^2 + 8x$

c) $4y + 10y^2$ **d)** $7y^3 + 14y^2$

7. Reflect Explain how to factor a polynomial using the GCF.

Tools

- TI-89 calculator

Method 2: Use a Computer Algebra System (CAS)

1. Clear the calculator's memory by selecting **2:NewProb** from the **Clean Up** menu.

2. Factor each expression. Press F2. Select **2:factor(**. Type the expression, and then press) ENTER. Record the results.

   a) $2x + 2$ **b)** $2x + 4$ **c)** $2x + 6$

   d) $2x + 8$ **e)** $2x + 10$

3. Consider the pairs of factors from step 2. How are they the same? How are they different? Provide a reason for your answers.

4. Use a CAS to factor each polynomial. Record the results.

   a) $3x^2 + 6x + 9$ **b)** $3x^2 + 12x + 15$

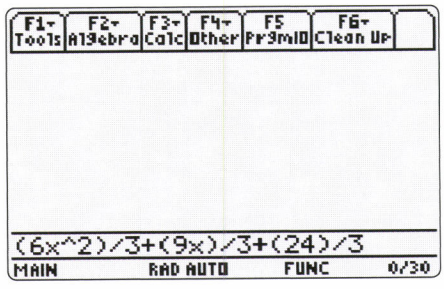
5. **a)** Look at the coefficients of the terms of each polynomial in step 4. What is the GCF?

   b) Look at the terms in the second factor from step 4. Since the first factor is the GCF of the polynomial, what operation provides the coefficients of the terms in the second factor?

6. **a)** Use a CAS to factor $6x^2 + 9x + 24$. Record the result.

   b) What is the GCF of the polynomial $6x^2 + 9x + 24$?

   c) Use a CAS to divide the GCF into each term of $6x^2 + 9x + 24$, as shown. Does the result match the second factor from part a)?



7. Use a CAS to factor each polynomial. Record the results.

   a) $2x^2 + 4x$ **b)** $2x^2 + 8x$

8. **a)** Look at the coefficients of the terms of each polynomial in step 7. What is the GCF? Look at the variable parts. What is the GCF?

   b) The first factor of each result from step 7 is the GCF of the polynomial. How is it related to your answers for part a)?

   c) Look at the terms in the second factor from step 7. Since the first factor is the GCF of the polynomial, what operation provides the terms in the second factor?

9. **a)** Use a CAS to factor $6x^2 + 15x$. Record the result.

   b) What is the GCF of the polynomial $6x^2 + 15x$?

   c) Use a CAS to divide the GCF into each term of $6x^2 + 15x$. Does the result match the second factor from part a)?

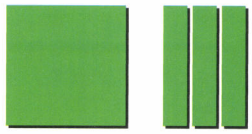
10. **Reflect** Explain how to factor a polynomial using the GCF.

A polynomial is factored when it is written as a product of two or more polynomials. Factoring a polynomial is the reverse process of expanding. To factor a polynomial:

- Find the GCF of the terms.
- Write the GCF as the first factor outside a set of brackets.
- Divide each term by the GCF, writing the result inside the brackets.

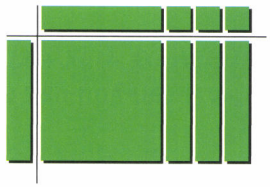
Example 1 Use a Model

Use algebra tiles to factor $x^2 + 3x$.



Solution

The polynomial $x^2 + 3x$ can be represented by a rectangle with area $x^2 + 3x$. The width of the rectangle is x and the length is $x + 3$. The dimensions of the rectangle are the factors of the polynomial.



$x^2 + 3x = x(x + 3)$

Example 2 Monomial Common Factor

Factor fully, if possible.

a) $6x + 3$

b) $8x^2 - 7x$

c) $25k^6 + 15k^4$

d) $21c^4d^3 - 28c^2d^5 + 7cd^3$

e) $5x^5y^3 + 7w^5z^2$

Solution

a) The GCF of the coefficients, 6 and 3, is 3.
There is no GCF of the variable parts.
Therefore, the GCF of the polynomial is 3.
Divide each term by 3.

$$6x + 3 = 3\left(\frac{6x}{3} + \frac{3}{3}\right)$$
$$= 3(2x + 1)$$

b) There is no common factor of the coefficients.
The GCF of the variable parts, x^2 and x, is x.
Therefore, the GCF of the polynomial is x.
Divide each term by x.

$$8x^2 - 7x = x\left(\frac{8x^2}{x} - \frac{7x}{x}\right)$$
$$= x(8x - 7)$$

c) The GCF of the coefficients, 25 and 15, is 5.
The GCF of the variable parts, k^6 and k^4, is k^4.
Therefore, the GCF of the polynomial is $5k^4$.
Divide each term by $5k^4$.

$$25k^6 + 15k^4 = 5k^4\left(\frac{25k^6}{5k^4} + \frac{15k^4}{5k^4}\right)$$

$$= 5k^4(5k^2 + 3)$$

When I compare the exponents, the GCF of the variable parts is the variable with the least exponent.

Apply the exponent laws.

d) The GCF of the coefficients, 21, -28, and 7, is 7.
The GCF of the variable parts, c^4d^3, c^2d^5 and cd^3, is cd^3.
Therefore, the GCF of the polynomial is $7cd^3$.
Divide each term by $7cd^3$ mentally.
$21c^4d^3 - 28c^2d^5 + 7cd^3 = 7cd^3(3c^3 - 4cd^2 + 1)$

I can check my answer by expanding.

$7cd^3(3c^3 - 4cd^2 + 1)$
$= 21c^4d^3 - 28c^2d^5 + 7cd^3$

e) Since the GCF of the terms of the polynomial $5x^5y^3 + 7w^5z^2$ is 1, it is not factorable.

Example 3 Binomial Common Factor

Factor.

a) $3x(y + 1) + 7z(y + 1)$

b) $2x(x - 3) - 5(x - 3)$

Solution

a) Think of $(y + 1)$ as one factor.
The GCF is the binomial $(y + 1)$.
Divide each term by $(y + 1)$ mentally.

$3x(y + 1) + 7z(y + 1)$
$= (y + 1)(3x + 7z)$

Write the GCF first. Collect the $(3x + 7z)$ in the second set of brackets.

b) Think of $(x - 3)$ as one factor.
The GCF is the binomial $(x - 3)$.
Divide each term by $(x - 3)$ mentally.

$2x(x - 3) - 5(x - 3)$
$= (x - 3)(2x - 5)$

Often there is no common factor for all the terms in a polynomial, but some of the terms have a common factor. A process of factoring by grouping can sometimes be used with these polynomials. This process involves factoring groups of terms first, instead of factoring the entire polynomial.

Example 4 Factor by Grouping

Factor.

a) $ax + ay + 2x + 2y$

b) $9x^2 + 15x + 3x + 5$

Solution

a) Group terms with a common factor. Factor the GCF from each grouping. Then, remove the binomial common factor.

$ax + ay + 2x + 2y$
$= (ax + ay) + (2x + 2y)$
$= a(x + y) + 2(x + y)$
$= (x + y)(a + 2)$

or

$ax + ay + 2x + 2y$
$= (ax + 2x) + (ay + 2y)$
$= x(a + 2) + y(a + 2)$
$= (a + 2)(x + y)$

b) $9x^2 + 15x + 3x + 5$
$= (9x^2 + 15x) + (3x + 5)$
$= 3x(3x + 5x) + 1(3x + 5)$
$= (3x + 5)(3x + 1)$

or

$9x^2 + 15x + 3x + 5$
$= (9x^2 + 3x) + (15x + 5)$
$= 3x(3x + 1) + 5(3x + 1)$
$= (3x + 1)(3x + 5)$

Key Concepts

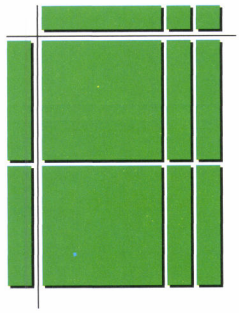
- Factoring a polynomial is the opposite of expanding a polynomial.

Factoring $\longrightarrow$
$x^2 + 3x = x(x + 3)$
$\longleftarrow$ **Expanding**

- To find the GCF of a polynomial, find the GCF of the coefficients, and then find the GCF of the variable parts.
- To factor a polynomial, remove the GCF as the first factor, and then divide each term by the GCF to obtain the second factor.
  $8x^2y^3 - 12x^4y = 4x^2y(2y^2 - 3x^2)$
- For polynomials with more than one variable, the GCF of the variable parts is the product of the common bases with the least exponent.
  The GCF of $2x^3y^4z^2 + 4x^2y^2z^3$ is $2x^2y^2z^2$.
- A common factor is not necessarily a monomial.
  $a(x + 2) + b(x + 2)$ has a binomial common factor of $(x + 2)$.
- To factor by grouping, factor groups of two terms with a common factor to produce a binomial common factor.

$bx + 3x + by + 3y = (bx + 3x) + (by + 3y)$
$= x(b + 3) + y(b + 3)$
$= (b + 3)(x + y)$

Communicate Your Understanding



C1 Explain how the diagram illustrates the factoring of a polynomial.

C2 Each of the following is an example of a common error when factoring. Describe each error and make the appropriate correction.

a) $35x^2 - 5x = 5x(7x - 0)$

b) $4y^3 + 7y^2 = 4y^2(y + 7)$

c) $16k^3m^2 - 8k^2m = 4k^2m(4km - 2)$

d) $9a^3b^5 + 6a^2b^4 = 3ab(3a^2b^4 + 2ab^3)$

C3 Describe how you would factor each polynomial.

a) $3c(d - 5) - 8(d - 5)$

b) $10x^2 - 14xy - 15x + 21y$

Practise

1. Find the GCF of each pair of terms.

a) $2x$ and $3x$ **b)** $6ab$ and $-8ac$

c) x^2 and x^3 **d)** k^4 and k^7

e) $3m^2$ and $5m$ **f)** $-12y^2$ and $-15y^4$

For help with question 2, see Example 1.

2. Use algebra tiles or a diagram to illustrate the factoring of each polynomial.

a) $x^2 + 5x$ **b)** $3x^2 + 6x$

c) $6x^2 + 4x$

For help with questions 3 and 4, see Example 2.

3. Factor fully, if possible.

a) $15w + 25z$ **b)** $3a - 11b$

c) $17ca - 8cd$ **d)** $9y - 8y^3$

e) $12b^4 + 18b^2$ **f)** $4g^2 - 8g + 6$

g) $7h + 3m - 5k$ **h)** $2n^5 + 12n^4 - 6n^3$

4. Factor fully, if possible.

a) $14x^2y + 16xy^3$ **b)** $10k^3m^2 - 6k^2m^2$

c) $8s^2y + 11t^3$ **d)** $66c^4de^2 - 22c^2de^2$

e) $7gh + 2mn - 13pq$

f) $5fg^2 - 25fg + 20f^2g$

g) $27r^2s^2 - 18r^3s^2 - 36rs^3$

h) $4n^2p^3 + 10n^4p^2 - 12n^3p^2$

For help with question 5, see Example 3.

5. Factor, if possible.

a) $3x(x + 8) + 5(x + 8)$

b) $a(b + 1) + 9c(b + 1)$

c) $2y(x - 5) + 4(x + 5)$

d) $4s(r + u) - t(r + u)$

For help with question 6, see Example 4.

6. Factor by grouping.

a) $mx + my + 2x + 2y$

b) $x^2 + 3x + 2x + 6$

c) $ay^2 + 3ay + 4y + 12$

d) $6x^2 + 9x - 2x - 3$

e) $16v^2 - 12v - 12v + 9$

Connect and Apply

7. a) Write a polynomial with two terms that has a GCF of 6.

b) Write a polynomial with three terms that has a GCF of x.

c) Write a polynomial with two terms that has a GCF of $5y^2$.

d) Write a polynomial with three terms that has a GCF of $2a^2b^3$.

8. The formula for the perimeter of a rectangle is $P = 2l + 2w$.

a) Write the formula in factored form.

b) If l represents 15 cm and w represents 9 cm, find the perimeter using both the original and the factored forms. What do you notice? Explain why this is so.

9. The formula for the surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$.

a) Write the formula in factored form.

b) If r represents 3 cm and h represents 8 cm, find the surface area using both the original and the factored forms. What do you notice? Explain why this is so.

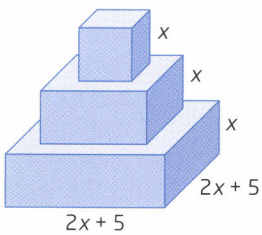
10. A rectangle has area given by the expression $6x^2 + 9x$. The length and width can be found by factoring the expression. Find all possible expressions for the length and width.

11. Binomial factors can differ by a factor of -1. An example is $7 - y$ and $y - 7$, since $7 - y$ can be rewritten as $-1(y - 7)$. Use this fact to factor each expression.

a) $5x(7 - y) + 4(y - 7)$

b) $5y(x - 1) + 2(1 - x)$

12. Chapter Problem The base length of each square-based prism in the pedestal design is 3 cm less than that of the layer immediately below.



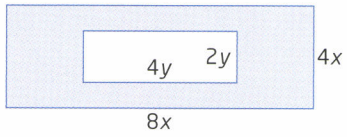
a) Write an algebraic expression for the total of the top surface areas of the three prisms used to make the pedestal.

b) Expand and simplify.

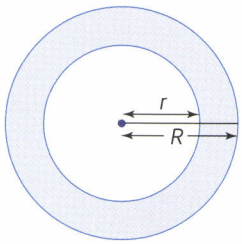
c) Factor the resulting expression from part b).

13. Write an expression, in fully factored form, for each of the shaded regions.

a)



b)



Extend

14. Factor the quadratic relation $y = 2x^2 - 3x$ to find the x-intercepts.

15. Factor each polynomial using a fraction as one of the common factors. Explain how this can simplify the operations when the values of the variables are known.

a) $\frac{1}{2}x^2 + \frac{3}{2}y^2$

b) $\frac{2}{3}a^3 - \frac{1}{3}ab$

c) $\frac{1}{6}k^4m^2 - \frac{1}{2}km^3 + \frac{1}{3}k^2m^2$

16. Math Contest If $3a + 8b = 12$, then what is the value of $15a + 40b$?

A 36

B 48

C 60

D 84

E 180

17. Math Contest Show that the sum of the squares of any five consecutive integers is divisible by 5.

5.4 Factor Quadratic Expressions of the Form $x^2 + bx + c$

A water garden combines a pond with aquatic plants and often ornamental fish, such as koi, to add visual appeal to the landscape. The area of a rectangular water garden can be represented by the **quadratic expression** $x^2 + 5x + 6$. To find the length and the width of the rectangle, you can write the trinomial as the product of two binomials.

quadratic expression

- a second-degree polynomial
- $4x^2 + 20$ and $x^2 + 7x + 10$ are quadratic expressions

Investigate A

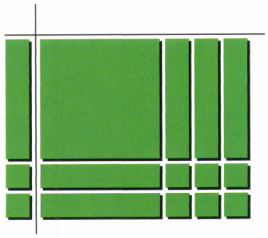
Tools

- algebra tiles

How can you use a model to factor quadratic expressions of the form $x^2 + bx + c$?

1. To factor $x^2 + 5x + 6$, use algebra tiles to create a rectangular area whose length and width represent the factors of the trinomial.

a) Arrange one x^2-tile, five x-tiles, and six unit tiles to form a rectangle with area $x^2 + 5x + 6$. Place tiles along the left side and top to find the length and width of the rectangle. One dimension has been done for you.



b) Write the equation for the trinomial as a product of the binomial dimensions.

2. Repeat step 1 for each trinomial.

a) $x^2 + 6x + 5$ **b)** $x^2 + 3x + 2$

c) $x^2 + 4x + 3$ **d)** $x^2 + 6x + 8$

3. Each trinomial in steps 1 and 2 is of the form $x^2 + bx + c$. What do you notice about b and c and the binomial factors for each trinomial? Describe the relationship.

4. Test your conclusions from step 3 on each trinomial. Use algebra tiles to check your answer.

a) $x^2 + 7x + 6$ **b)** $x^2 + 8x + 12$

5. **Reflect** Describe a process for finding the factors of a quadratic expression of the form $x^2 + bx + c$.

Investigate B

How can you use patterns to factor quadratic expressions of the form $x^2 + bx + c$?

A: Positive Values of b and c

1. Expand and simplify each product. Try to apply the distributive property mentally.
   - **a)** $(x + 4)(x + 3)$
   - **b)** $(x + 1)(x + 5)$
   - **c)** $(x + 7)(x + 8)$

2. The result of expanding each binomial product of the form $(x + r)(x + s)$ in step 1 is a trinomial of the form $x^2 + bx + c$. Describe how you calculated b and c using the values of r and s.

3. Use the patterns from step 2 to reverse the process. Write each trinomial of the form $x^2 + bx + c$ as a binomial product of the form $(x + r)(x + s)$.
   - **a)** $x^2 + 6x + 8$
   - **b)** $x^2 + 7x + 10$
   - **c)** $x^2 + 9x + 20$
   - **d)** $x^2 + 10x + 21$

4. **Reflect** Describe a process for factoring quadratic expressions of the form $x^2 + bx + c$.

B: Negative Values of b and/or c

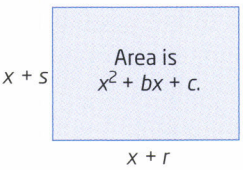
1. Expand and simplify each product. Try to apply the distributive property mentally.
   - **a)** $(x - 3)(x - 2)$
   - **b)** $(x - 1)(x - 5)$
   - **c)** $(x - 1)(x + 5)$
   - **d)** $(x + 3)(x - 8)$

2. The result of expanding each binomial product of the form $(x + r)(x + s)$ in step 1 is a trinomial of the form $x^2 + bx + c$.
   - **a)** Describe how you determined the signs of the values of b and c when both values of r and s were negative.
   - **b)** Describe how you determined the signs of the values of b and c when only one of the values of r and s was negative.

3. Use your process to factor each trinomial.
   - **a)** $x^2 - 7x + 10$
   - **b)** $x^2 + 4x - 5$
   - **c)** $x^2 - 4x - 5$
   - **d)** $x^2 - 3x - 10$

4. **Reflect** How does your process for factoring quadratic expressions of the form $x^2 + bx + c$ change when the values of b and/or c are negative?

By finding the dimensions of a rectangle whose area is a quadratic expression, you are reversing the process of expanding two binomials that you learned in Section 5.1. This process is called factoring.



Another way to factor a quadratic expression of the form $x^2 + bx + c$ is to study the patterns from multiplying two binomials.

$$\begin{aligned}(x + r)(x + s) &= x^2 + sx + rx + rs \\ &= x^2 + rx + sx + rs \\ &= x^2 + (r + s)x + rs\end{aligned}$$

Therefore, $x^2 + bx + c = (x + r)(x + s)$, where $r + s = b$ and $r \times s = c$.

In general, you will factor *over the integers*, meaning that the values of r and s are integers only.

Many quadratic expressions, such as $x^2 + 3x + 5$, cannot be factored over the integers. No two integers have a product of 5 and a sum of 3.

Example 1 Factor Quadratic Expressions

Factor, if possible.

a) $x^2 + 7x + 12$

b) $x^2 + 4x + 6$

c) $x^2 - 29x + 28$

d) $x^2 + 3x - 18$

e) $x^2 - 4x - 21$

Solution

a) For $x^2 + 7x + 12$, $b = 7$ and $c = 12$. Use a table to find two integers whose product is 12 and whose sum is 7. In order to have a positive product and a positive sum, both numbers must be positive.

Factors of 12	Product	Sum
1, 12	12	13
2, 6	12	8
3, 4	12	7

Therefore, r is 3 and s is 4.
$x^2 + 7x + 12 = (x + 3)(x + 4)$

b) For $x^2 + 4x + 6$, $b = 4$ and $c = 6$.

Factors of 6	Product	Sum
1, 6	6	7
2, 3	6	5

I need to find two positive integers whose product is 6 and whose sum is 4.

Since no two integers have a product of 6 and sum of 4, $x^2 + 4x + 6$ cannot be factored over the integers.

c) For $x^2 - 29x + 28$, $b = -29$ and $c = 28$.

Factors of 28	Product	Sum
−1, −28	28	−29
−2, −14	28	−16
−4, −7	28	−11

I need to find two integers whose product is 28 and whose sum is −29. To have a positive product and a negative sum, both numbers must be negative.

Therefore, r is -1 and s is -28.
$x^2 - 29x + 28 = (x - 1)(x - 28)$

d) For $x^2 + 3x - 18$, $b = 3$ and $c = -18$.

Factors of −18	Product	Sum
1, −18	−18	−17
−1, 18	−18	17
2, −9	−18	−7
−2, 9	−18	7
3, −6	−18	−3
−3, 6	−18	3

I need to find two integers whose product is −18 and whose sum is 3. To have a negative product and a positive sum, one number must be negative and the other positive.

Therefore, r is -3 and s is 6.
$x^2 + 3x - 18 = (x + 6)(x - 3)$

e) For $x^2 - 4x - 21$, $b = -4$ and $c = -21$.

Factors of −21	Product	Sum
1, −21	−21	−20
−1, 21	−21	20
3, −7	−21	−4
−3, 7	−21	4

I need to find two integers whose product is −21 and whose sum is −4. To have a negative product and a negative sum, one number must be negative and the other positive.

Therefore, r is 3 and s is -7.
$x^2 - 4x - 21 = (x + 3)(x - 7)$

Example 2 Dimensions of a Water Garden

a) Determine binomials that represent the dimensions of the rectangular water garden.

b) Determine the dimensions if x represents 1 m.

Area is $x^2 + 5x + 6$.

Solution

a) Factor the quadratic expression for the area. Find two integers whose product is 6 and whose sum is 5. The integers are 2 and 3.
$x^2 + 5x + 6 = (x + 2)(x + 3)$
The dimensions can be represented by $x + 2$ and $x + 3$.

b) Substitute $x = 1$ into $x + 2$ and $x + 3$.

$1 + 2 = 3$

$1 + 3 = 4$

The water garden has dimensions of 3 m by 4 m.

Key Concepts

- To factor a quadratic expression of the form $x^2 + bx + c$, first find two integers whose product is c and whose sum is b.
  - For $x^2 + 12x + 27$, find two integers whose product is 27 and whose sum is 12. The integers are 3 and 9.
  - Express the quadratic expression as a product, $x^2 + bx + c = (x + r)(x + s)$.

    $x^2 + 12x + 27 = (x + 3)(x + 9)$
- Not all quadratic expressions of the form $x^2 + bx + c$ can be factored over the integers.

Communicate Your Understanding

C1 Describe how to use algebra tiles to factor $x^2 + 8x + 7$.

C2 Explain the steps you would use to factor $y^2 - 6y - 40$.

C3 Explain why $k^2 + 5k - 9$ cannot be factored over the integers.

Practise

1. Illustrate the factoring of each trinomial using algebra tiles or a diagram.

a) $x^2 + 4x + 3$ **b)** $x^2 + 7x + 10$

c) $x^2 + 6x + 8$ **d)** $x^2 + 4x + 4$

2. Find two integers with the given product and sum.

a) product = 45, sum = 14

b) product = 6, sum = −5

c) product = −10, sum = 3

d) product = −20, sum = −8

For help with questions 3 to 5, see Example 1.

3. Factor, if possible.

a) $x^2 + 7x + 10$ **b)** $j^2 + 12j + 27$

c) $k^2 + 5k + 4$ **d)** $p^2 + 9p + 12$

e) $w^2 + 11w + 25$ **f)** $d^2 + 10d + 24$

4. Factor, if possible.

a) $m^2 - 7m + 10$ **b)** $x^2 - 5x + 7$

c) $y^2 - 5y + 4$ **d)** $r^2 - 16r + 64$

e) $w^2 - 9w + 24$ **f)** $q^2 - 10q + 9$

5. Factor, if possible.

a) $a^2 - 3a - 10$ **b)** $s^2 + 3s - 10$

c) $d^2 - 8d - 9$ **d)** $f^2 + 7f - 6$

e) $g^2 - 5g - 14$ **f)** $r^2 + 2r - 6$

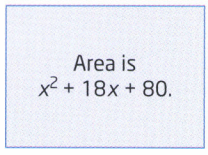
g) $x^2 + x - 42$ **h)** $b^2 - 2b - 4$

Connect and Apply

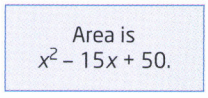
For help with question 6, see Example 2.

6. Determine binomials that represent the length and width of each rectangle. Then, determine the dimensions of the rectangle if x represents 15 cm.

a)



b)



7. Factor completely by first removing the greatest common factor (GCF).

a) $3x^2 + 12x + 9$

b) $2d^2 - 22d + 56$

c) $5z^2 + 40z + 60$

d) $4s^2 - 8s - 32$

e) $bx^2 + 10bx - 24b$

f) $x^3 + 18x^2 + 72x$

8. Determine two values of b so that each expression can be factored.

a) $x^2 + bx + 12$ **b)** $x^2 - bx + 4$

c) $x^2 - bx - 8$ **d)** $x^2 + bx - 10$

9. Determine two values of c so that each expression can be factored.

a) $x^2 + 6x + c$ **b)** $x^2 - x + c$

c) $x^2 - 8x - c$ **d)** $x^2 + 2x - c$

10. Expand each pair of binomials. Compare the answers.

a) $(x + 1)(x + 3)$ and $(x + y)(x + 3y)$

b) $(x + 4)(x - 6)$ and $(x + 4y)(x - 6y)$

c) $(x - 2)(x + 9)$ and $(x - 2y)(x + 9y)$

d) $(x - 6)(x - 9)$ and $(x - 6y)(x - 9y)$

11. Factor. How does the additional variable change your thinking?

a) $a^2 + 11ab + 24b^2$

b) $k^2 - 11km + 18m^2$

c) $c^2 + 4cd - 21d^2$

d) $x^2 - 6xy - 16y^2$

12. a) Make up an example of a quadratic expression that cannot be factored.

b) Explain why it cannot be factored.

13. A parabola has equation $y = x^2 - 4x - 12$.

a) Factor the right side of the equation.

b) Identify the x-intercepts of the parabola.

c) Find the equation of the axis of symmetry, find the vertex, and draw the graph.

14. The height of a ball thrown from the top of a building can be approximated by the formula $h = -5t^2 + 15t + 20$, where t is the time, in seconds, and h is the height, in metres.

a) Write the formula in factored form. Hint: Remove the GCF first.

b) How can you use the factors to find when the ball lands on the ground?

Extend

15. a) Explain how $x^4 + 9x^2 + 20$ and $x^2 + 9x + 20$ are alike. How are they different?

b) Factor $x^4 + 9x^2 + 20$.

16. Refer to question 15. Factor.

a) $x^4 + 11x^2 + 30$

b) $x^4 - 7x^2y^2 + 12y^2$

c) $x^6 - 3x^3 - 54$

d) $3(x - 5)^2 + 27(x - 5) - 66$

17. Math Contest

a) Expand $(x + 2)^3$.

b) Factor $x^3 + 9x^2 + 27x + 27$.

c) Factor $8a^3 + 60a^2b + 150ab^2 + 125b^3$.

5.5 Factor Quadratic Expressions of the Form $ax^2 + bx + c$

The Ontario Summer Games are held every two years in even-numbered years to provide sports competition for youth between the ages of 11 and 22. At the Games, approximately 2500 athletes from across the province compete in 19 sports.

Beach volleyball is one of the sports on the Games program. It is played by two teams of two players on a sand court with area given by $3x^2 + 10x + 3$. Algebraic expressions for the dimensions of the court can be found by factoring the trinomial expression.

Investigate

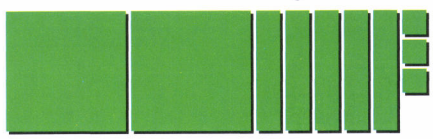
Tools

- algebra tiles

How can you use a model to factor quadratic expressions of the form $ax^2 + bx + c$?

1. Use algebra tiles to form a rectangle to model the product $(2x + 1)(x + 2)$.

2. Arrange the algebra tiles representing the trinomial $2x^2 + 5x + 3$ to form a rectangle. Identify the binomials that represent the length and width of the rectangle.



3. Repeat step 2 for each trinomial.

   a) $3x^2 + 5x + 2$ **b)** $4x^2 + 8x + 3$

4. Each trinomial represents the area of a rectangle. Draw diagrams and identify the binomials that represent the length and width of the rectangle.

   a) $2x^2 + 5x + 2$ **b)** $5x^2 + 8x + 3$

5. **Reflect** Describe how to use algebra tiles to factor a quadratic trinomial of the form $ax^2 + bx + c$.

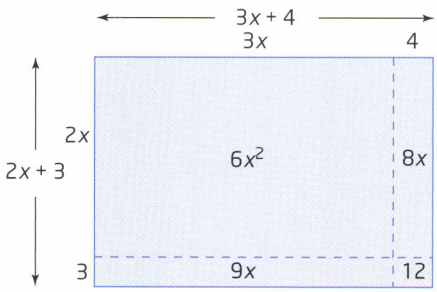
6. **Reflect** Can you see a way to factor trinomials of the form $ax^2 + bx + c$ without using algebra tiles? If so, describe it.

When you expand two binomials, you add the two middle terms.

$$(2x + 3)(3x + 4) = 6x^2 + 8x + 9x + 12$$
$$= 6x^2 + 17x + 12$$

Notice the following patterns.
$8 + 9 = 17$ and $8 \times 9 = 6 \times 12$

You can use these patterns and the method of factoring by grouping to factor trinomials of the form $ax^2 + bx + c$. Work in reverse by replacing the middle term with two terms whose integer coefficients have a product of $a \times c$ and a sum of b.



Example 1 Break up the Middle Term

Factor, if possible.

a) $3x^2 + 8x + 4$

b) $3x^2 + 2x + 4$

c) $6x^2 - 5x + 1$

Solution

a) For $3x^2 + 8x + 4$, $a = 3$, $b = 8$, and $c = 4$.
Use a table to find two integers whose product is 3×4, or 12, and whose sum is 8. In order to have a positive product and a positive sum, both integers must be positive.

Factors of 12	Product	Sum
1, 12	12	13
2, 6	12	8
3, 4	12	7

Since the integers 2 and 6 satisfy this product and sum, break up $8x$ into $2x + 6x$. Then, factor by grouping.

$$3x^2 + 8x + 4$$
$$= 3x^2 + 2x + 6x + 4$$
$$= (3x^2 + 2x) + (6x + 4)$$
$$= x(3x + 2) + 2(3x + 2)$$
$$= (3x + 2)(x + 2)$$

Literacy Connections

Since *break up* and *decompose* mean the same thing, the method of breaking up the middle term is sometimes referred to as the decomposition method.

b) For $3x^2 + 2x + 4$, $a = 3$, $b = 2$, and $c = 4$.
Since there is no pair of integers that satisfy these conditions, $3x^2 + 2x + 4$ is not factorable over the integers.

Factors of 12	Product	Sum
1, 12	12	13
2, 6	12	8
3, 4	12	7

I need to find two integers whose product is 3×4, or 12, and whose sum is 2. Since the product and the sum are positive, I need two positive integers.

c) For $6x^2 - 5x + 1$, $a = 6$, $b = -5$, and $c = 1$.
Since the integers -2 and -3 satisfy this product and sum, break up $-5x$ into $-2x - 3x$. Then, factor by grouping.

$$\begin{aligned} & 6x^2 - 5x + 1 \\ = \ & 6x^2 - 2x - 3x + 1 \\ = \ & (6x^2 - 2x) + (-3x + 1) \\ = \ & 2x(3x - 1) - 1(3x - 1) \\ = \ & (3x - 1)(2x - 1) \end{aligned}$$

Factors of 6	Product	Sum
–1, –6	6	–7
–2, –3	6	–5

I need to find two integers whose product is 6×1, or 6, and whose sum is –5. Since the product is positive and the sum is negative, I need two negative integers.

Example 2 Trinomials With Two Variables

Factor $10x^2 - 3xy - 4y^2$.

Solution

For $10x^2 - 3xy - 4y^2$, $a = 10$, $b = -3$, and $c = -4$.

I need to find two integers whose product is $10 \times (-4)$, or –40, and whose sum is –3. The integers 5 and –8 work.

Factors of –40	Product	Sum
1, –40	–40	–39
2, –20	–40	–18
4, –10	–40	–6
5, –8	–40	–3
–1, 40	–40	39
–2, 20	–40	18
–4, 10	–40	6
–5, 8	–40	3

$$\begin{aligned} & 10x^2 - 3xy - 4y^2 \\ = \ & 10x^2 + 5xy - 8xy - 4y^2 \\ = \ & (10x^2 + 5xy) + (-8xy - 4y^2) \\ = \ & 5x(2x + y) - 4y(2x + y) \\ = \ & (2x + y)(5x - 4y) \end{aligned}$$

Break up $-3xy$ into $5xy - 8xy$.
Factor by grouping.

Example 3 Remove a Common Factor

Factor $16x^2 + 26x - 12$.

Solution

First, remove the greatest common factor (GCF), and then proceed as before.

The GCF of the polynomial $16x^2 + 26x - 12$ is 2.

$$16x^2 + 26x - 12$$
$$= 2(8x^2 + 13x - 6)$$

To factor $8x^2 + 13x - 6$, I need to find two integers whose product is $8 \times (-6)$, or -48, and whose sum is 13. The integers -3 and 16 work.

Factors of −48	Product	Sum
−1, 48	−48	47
−2, 24	−48	22
−3, 16	−48	13
−4, 12	−48	8
−6, 8	−48	2
1, −48	−48	−47
2, −24	−48	−22
3, −16	−48	−13
4, −12	−48	−8
6, −8	−48	−2

$$\begin{aligned} 16x^2 + 26x - 12 &= 2(8x^2 + 13x - 6) \\ &= 2(8x^2 - 3x + 16x - 6) && \text{Break up } 13x \text{ into } -3x + 16x. \\ &= 2[(8x^2 - 3x) + (16x - 6)] && \text{Factor by grouping.} \\ &= 2[x(8x - 3) + 2(8x - 3)] \\ &= 2[(8x - 3)(x + 2)] \\ &= 2(8x - 3)(x + 2) \end{aligned}$$

Key Concepts

- Always look for a common factor first when factoring a trinomial.
- To factor $ax^2 + bx + c$, find two integers whose product is $a \times c$ and whose sum is b. Then, break up the middle term and factor by grouping.
- Not all quadratic expressions of the form $ax^2 + bx + c$ can be factored over the integers.

Communicate Your Understanding

C1 When you use algebra tiles to factor a trinomial, why do you need to be able to form a rectangle with the tiles?

C2 When factored, $2x^2 + 9x + 9$ can be written as $(2x + 3)(x + 3)$. Can it also be written as $(x + 3)(2x + 3)$? Justify your answer using words and a diagram.

C3 Describe how you would factor $5x^2 + 18x + 9$.

Practise

1. Use algebra tiles or a diagram to factor each trinomial.

a) $2x^2 + 5x + 3$

b) $3x^2 + 7x + 4$

c) $6x^2 + 5x + 1$

d) $6x^2 + 11x + 4$

For help with questions 2 to 4, see Example 1.

2. Factor, if possible.

a) $2x^2 + 7x + 5$

b) $6y^2 + 19y + 8$

c) $4k^2 + 15k + 9$

d) $3m^2 + 10m + 8$

e) $10w^2 + 15w + 3$

f) $12q^2 + 17q + 6$

3. Factor, if possible.

a) $4x^2 - 11x + 6$

b) $5n^2 - 11n + 6$

c) $6c^2 - 3c + 1$

d) $6a^2 - 7a + 1$

e) $9b^2 - 24b + 7$

f) $15k^2 - 19k + 6$

4. Factor, if possible.

a) $3y^2 + 4y - 7$

b) $2m^2 + 3m - 9$

c) $8k^2 - 6k - 5$

d) $12y^2 + y - 1$

e) $9x^2 - 15x - 4$

f) $5h^2 - 14h - 3$

For help with question 5, see Example 2.

5. Factor.

a) $3x^2 + 7xy + 2y^2$

b) $6m^2 + 13mn + 2n^2$

c) $2p^2 - 11pq + 5q^2$

d) $6c^2 - 7cd - 10d^2$

e) $9x^2 - 9xy - 4y^2$

f) $6d^2 + de - 2e^2$

For help with question 6, see Example 3.

6. Factor.

a) $8k^2 - 16k + 6$

b) $9p^2 + 15p - 6$

c) $6m^2 - 14m - 12$

d) $10x^2 + 15x - 10$

e) $10r^2 - 22r + 4$

f) $8y^2 - 22y + 12$

Connect and Apply

7. Factor. Then, substitute $x = 2$ into both forms. Are the results the same? Explain.

a) $4x^2 + 12x + 5$

b) $7x^2 - 23x + 6$

c) $15x^2 - 2x - 8$

d) $8x^2 + 14x - 4$

e) $6x^2 - 19x + 15$

f) $5x^2 + 18x + 9$

8. Find two values of n so that each trinomial can be factored over the integers.

a) $x^2 + nx + 16$

b) $3y^2 + ny + 25$

c) $6a^2 + nab + 7b^2$

9. Find two values of k so that each trinomial can be factored over the integers.

a) $36m^2 + 8m + k$

b) $18y^2 - 42y + k$

c) $kp^2 - 72pq + 16q^2$

10. Describe the steps in determining whether you can factor $ax^2 + bx + c$ over the integers.

11. Explain why it is easier to factor $ax^2 + bx + c$ if a and c are prime numbers.

12. A rectangle has area defined by $6x^2 + 13x - 8$.

Area is
$6x^2 + 13x - 8$.

a) Factor to find algebraic expressions for the length and width of the rectangle.

b) If x represents 10 cm, determine the perimeter and area of the rectangle.

13. The height, h, in metres, of a toy rocket at any time, t, in seconds, during its flight can be estimated using the formula $h = -5t^2 + 23t + 10$. Write the formula in factored form and determine when the rocket will fall to the ground.

14. Use Technology The range, r, in kilometres, of an airplane with full tanks at a power setting of p revolutions per minute (RPM) can be modelled by the relation $r = -0.0008p^2 + 3.2p - 2400$.

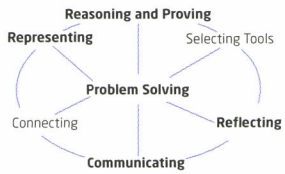
a) Use a computer algebra system to factor the trinomial.

b) Describe the set of values that p may take for this model.

c) Determine what value of p results in the maximum range.

15. The total revenue from sales of ski jackets is modelled by the expression $720 + 4x - 2x^2$. Revenue is also calculated as the product of the number of jackets sold and the price per jacket. Determine expressions for the number sold and the price per jacket. Hint: As the price increases, the number sold decreases.

Achievement Check



16. a) Factor $5x^2 + 11x + 2$.

b) Write a quadratic trinomial that cannot be factored over the integers. Explain how you know.

c) The area of a square is $81 - 72x + 16x^2$. If x must be a positive integer, what is the least possible measure for the perimeter of the square?

Extend

17. Factor.

a) $5x^4 + 18x^2 + 9$

b) $7x^4 - 13x^2y^2 + 6y^4$

c) $6x^6 + 13x^3y^3 - 8y^6$

d) $10m^6 - 7m^3n^2 - 12n^4$

18. Factor.

a) $2(x + a)^2 + 3(x + a) + 1$

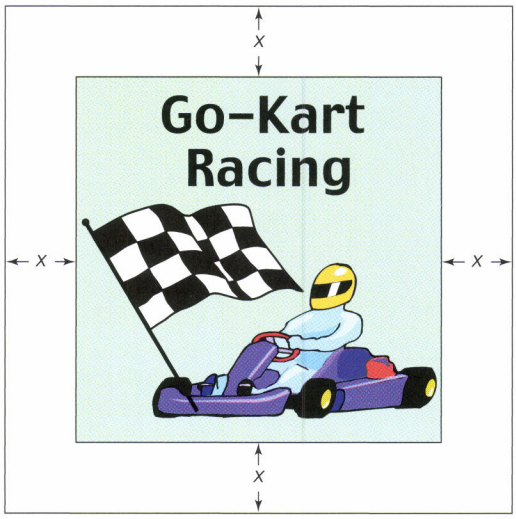
b) $2(x - b)^2 + 5(x - b) + 2$

19. a) A shape has area defined by $A = 8x^2 + 10x - 7$. Identify the shape(s).

b) A solid has volume defined by $V = 4x^3 - 12x^2y + 9xy^2$. Identify the type of solid.

5.6 Factor a Perfect Square Trinomial and a Difference of Squares

One of the sponsors for the school yearbook has asked that the area of the art in their advertisement be increased by the same amount on all sides. The expression for the area of the enlarged art is given by $4x^2 + 12x + 9$, which is a perfect square trinomial.



In Section 5.2, you learned that some polynomial products can be expanded using special patterns. Similarly, you can factor polynomials that are perfect square trinomials or differences of squares using special patterns.

Investigate A

How can you use patterns to factor a difference of squares?

Method 1: Use Pencil and Paper

1. Expand and simplify or use the pattern for the product of the sum and the difference of two terms from Section 5.2.
   - **a)** $(x + 1)(x - 1)$
   - **b)** $(y + 2)(y - 2)$
   - **c)** $(3c - 10)(3c + 10)$
   - **d)** $(2m - 4)(2m + 4)$
2. How are the two binomials being multiplied in step 1 alike? How are they different?
3. Consider each simplified expansion from step 1.
   - **a)** How is the first term related to the first terms of the two binomials?
   - **b)** How is the last term related to the last terms of the two binomials?
4. **Reflect** Each resulting product in step 1 is a difference of squares. Explain how you can identify a difference of squares.

5. Confirm that each polynomial is a difference of squares. Then, use the reverse process to factor each. Check by expanding and simplifying.
   a) $x^2 - 25$
   b) $y^2 - 36$
   c) $16k^2 - 49$
   d) $25n^2 - 144$

6. **Reflect** Write a rule for factoring a difference of squares.

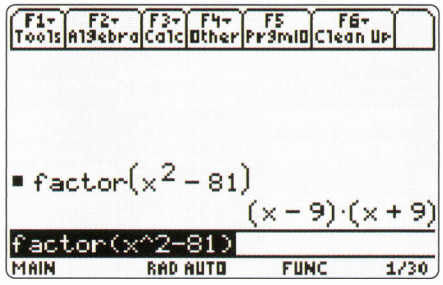
7. Use your rule to factor $100y^2 - 49x^2$. Check by expanding and simplifying.

Method 2: Use a Computer Algebra System (CAS)

Tools
- TI-89 calculator

1. Clear the calculator's memory by selecting **2:NewProb** from the **Clean Up** menu.

2. Use the **Factor** function on each polynomial. Record the results.
   a) $x^2 - 81$
   b) $y^2 - 64$
   c) $25d^2 - 36$
   d) $16k^2 - 121$
   e) $144b^2 - 25k^2$
   f) $4n^2 - 49p^2$



3. **Reflect** Each polynomial in step 2 is a difference of squares. Explain how you can identify a difference of squares.

4. Consider each pair of binomial factors from step 2.
   a) How are they alike? How are they different?
   b) How are the first terms of the factors related to the first term of the polynomial?
   c) How are the last terms of the factors related to the last term of the polynomial?

5. **Reflect** Write a rule for factoring a difference of squares.

Investigate B

How can you use patterns to factor a perfect square trinomial?

Method 1: Use Pencil and Paper

1. Expand and simplify or use the pattern for squaring a binomial from Section 5.2.
   a) $(x + 3)^2$
   b) $(y - 5)^2$
   c) $(k + 7)^2$
   d) $(2h + 3)^2$
   e) $(3b - 5)^2$

2. Consider each simplified expansion from step 1.
   a) How is the first term in each trinomial related to the first term in each binomial?
   b) How is the last term in each trinomial related to the last term in each binomial?
   c) How is the middle term in each trinomial related to the terms in the binomial?

3. **Reflect** Each resulting product in step 1 is a perfect square trinomial. Explain how you can identify a perfect square trinomial.

4. Confirm that each polynomial is a perfect square trinomial. Then, use the reverse process to factor each. Check by expanding and simplifying.
   a) $x^2 + 12x + 36$
   b) $y^2 - 6y + 9$
   c) $4k^2 + 20k + 25$
   d) $9k^2 - 24k + 16$

5. **Reflect** Write a rule for factoring perfect square trinomials.

6. Test your rule by factoring $x^2 + 14x + 49$. Check by expanding and simplifying.

Tools
- TI-89 calculator

Method 2: Use a CAS

1. Clear the calculator's memory by selecting **2:NewProb** from the **Clean Up** menu.

2. Use the **Factor** function on each polynomial. Record the results.
   a) $x^2 + 8x + 16$
   b) $y^2 - 10y + 25$
   c) $4k^2 - 20k + 25$
   d) $9k^2 + 24k + 16$
   e) $25t^2 + 30t + 9$
   f) $16z^2 - 8z + 1$

3. Consider each resulting square of a binomial from step 2.
   a) How is the first term of the binomial related to the first term of the trinomial?
   b) How is the last term of the binomial related to the last term of the trinomial?
   c) How are the terms of the binomial related to the middle term of the trinomial?

4. **Reflect** Each polynomial in step 2 is a perfect square trinomial. Explain how you can identify a perfect square trinomial.

5. **Reflect** Write a rule for factoring perfect square trinomials.

In Section 5.2, you saw that $(a + b)(a - b) = a^2 - b^2$. You can factor a difference of squares as $a^2 - b^2 = (a + b)(a - b)$.

You also saw that $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. You can factor a perfect square trinomial as $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$.

Example 1 Difference of Squares

Factor.

a) $x^2 - 100$ **b)** $98a^2 - 450b^2$

Solution

a) $a^2 - b^2 = (a + b)(a - b)$ Use the pattern for a difference of squares.

$x^2 - 100 = (x)^2 - 10^2$

$= (x + 10)(x - 10)$

b) $98a^2 - 450b^2 = 2(49a^2 - 225b^2)$ Remove the greatest common factor.

$= 2[(7a)^2 - (15b)^2]$

$= 2(7a + 15b)(7a - 15b)$ Factor the difference of squares.

Example 2 Perfect Square Trinomials

Verify that each trinomial is a perfect square. Then, factor.

a) $x^2 + 6x + 9$

b) $x^2 - 12x + 36$

Solution

a) Since $x^2 = (x)^2$ and $9 = 3^2$, the first and last terms are perfect squares.
Since $6x = 2(x)(3)$, the middle term is twice the product of the square roots of the first and last terms.
Therefore, $x^2 + 6x + 9$ is a perfect square trinomial.

$a^2 + 2ab + b^2 = (a + b)^2$ Use the appropriate perfect square trinomial pattern.

$x^2 + 6x + 9 = (x)^2 + 2(x)(3) + 3^2$

$= (x + 3)^2$

b) Since $x^2 = (x)^2$ and $36 = 6^2$, the first and last terms are perfect squares.
Twice the product of these square roots is $2(x)(6) = 12x$.
Therefore, $x^2 - 12x + 36$ is a perfect square trinomial.

$a^2 - 2ab + b^2 = (a - b)^2$ Use the appropriate perfect square trinomial pattern.

$x^2 - 12x + 36 = (x)^2 - 2(x)(6) + 6^2$

$= (x - 6)^2$

The middle term of the trinomial is $-12x$, so a difference has been squared.

Example 3 More Complex Perfect Square Trinomials

Verify that each trinomial is a perfect square. Then, factor.

a) $4x^2 + 28x + 49$

b) $25k^2 - 60km + 36m^2$

Solution

a) Since $4x^2 = (2x)^2$ and $49 = 7^2$, the first and last terms are perfect squares.
Since $28x = 2(2x)(7)$, the middle term is twice the product of the square roots of the first and last terms.
Therefore, $4x^2 + 28x + 49$ is a perfect square trinomial.

$$\begin{aligned} 4x^2 + 28x + 49 &= (2x)^2 + 2(2x)(7) + 7^2 \\ &= (2x + 7)^2 \end{aligned}$$

The middle term of the trinomial is $-60km$, so a difference has been squared.

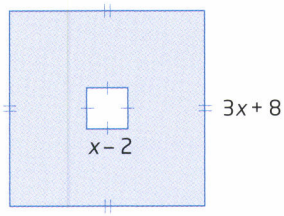
b) Since $25k^2 = (5k)^2$ and $36m^2 = (6m)^2$, the first and last terms are perfect squares.
Twice the product of these square roots is $2(5k)(6m) = 60km$.
Therefore, $25k^2 - 60km + 36m^2$ is a perfect square trinomial.

$$\begin{aligned} 25k^2 - 60km + 36m^2 &= (5k)^2 - 2(5k)(6m) + (6m)^2 \\ &= (5k - 6m)^2 \end{aligned}$$

Example 4 Area of a Region

a) Find an algebraic expression for the area of the shaded region.

b) Write the area expression in factored form.



Solution

a) The area of the shaded region is the difference in the areas of the two squares.
Area $= (3x + 8)^2 - (x - 2)^2$

b) Method 1: Expand, Then Factor

$$\begin{aligned} &(3x + 8)^2 - (x - 2)^2 \\ &= 9x^2 + 48x + 64 - (x^2 - 4x + 4) \\ &= 9x^2 + 48x + 64 - x^2 + 4x - 4 \\ &= 8x^2 + 52x + 60 \\ &= 4(2x^2 + 13x + 15) \\ &= 4(2x^2 + 10x + 3x + 15) \\ &= 4[(2x^2 + 10x) + (3x + 15)] \\ &= 4[2x(x + 5) + 3(x + 5)] \\ &= 4[(x + 5)(2x + 3)] \\ &= 4(x + 5)(2x + 3) \end{aligned}$$

Method 2: Factor as a Difference of Squares

This is a difference of squares, $a^2 - b^2$, with $a = (3x + 8)$ and $b = (x - 2)$.

$$\begin{aligned}
&(3x + 8)^2 - (x - 2)^2 \\
&= [(3x + 8) + (x - 2)][(3x + 8) - (x - 2)] \\
&= (3x + 8 + x - 2)(3x + 8 - x + 2) \\
&= (4x + 6)(2x + 10) \\
&= 2(2x + 3)[2(x + 5)] \\
&= 4(2x + 3)(x + 5)
\end{aligned}$$

Key Concepts

- Always look for a common factor first when factoring a trinomial.
- You can factor a difference of squares as $a^2 - b^2 = (a + b)(a - b)$.
- You can factor a perfect square trinomial as $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$.

Communicate Your Understanding

C1 Use words and diagrams to explain why $x^2 + 9$ cannot be factored over the integers.

C2 When her classmate showed Barbara the first step in Example 3b), $25k^2 - 60km + 36m^2 = (5k)^2 - 2(5k)(6m) + (6m)^2$, Barbara asked, "Where did the 2 come from?" Answer Barbara's question.

Practise

For help with questions 1 and 2, see Example 1.

1. Factor.

a) $x^2 - 16$ **b)** $y^2 - 100$

c) $9k^2 - 36$ **d)** $4a^2 - 121$

e) $36w^2 - 49$ **f)** $144p^2 - 1$

g) $16n^2 - 25$ **h)** $100g^2 - 81$

2. Factor.

a) $m^2 - 49n^2$ **b)** $h^2 - 25d^2$

c) $100 - 9c^2$ **d)** $169a^2 - 49b^2$

e) $25x^2 - 36y^2$ **f)** $16c^2 - 9d^2$

g) $162 - 8s^2$ **h)** $75h^2 - 27g^2$

For help with question 3, see Example 2.

3. Verify that each trinomial is a perfect square. Then, factor.

a) $x^2 + 12x + 36$ **b)** $k^2 + 18k + 81$

c) $y^2 - 6y + 9$ **d)** $m^2 - 14m + 49$

e) $x^2 + 20x + 100$ **f)** $64 - 16r + r^2$

For help with question 4, see Example 3.

4. Verify that each trinomial is a perfect square. Then, factor.

a) $4c^2 + 12c + 9$ **b)** $16k^2 - 8k + 1$

c) $25x^2 + 70x + 49$ **d)** $9y^2 - 30y + 25$

e) $100c^2 - 180c + 81$ **f)** $25 + 80y + 64y^2$

Connect and Apply

5. Each of the following is not factorable over the integers. Why not?

a) $9x^2 - 16y$

b) $36a^2 + 107a + 81$

c) $10w^2 - 70wz + 49z^2$

d) $25n^2 + 36m^2$

6. Factor fully, if possible.

a) $4x^2 + 28xy + 49y^2$

b) $9k^2 - 24km + 16m^2$

c) $25p^2 + 60pq + 144q^2$

d) $9y^2 - 7x^2$

e) $2a^2 - 28ab + 98b^2$

f) $196n^2 - 144m^2$

g) $25x^2 + 70xy + 14y^2$

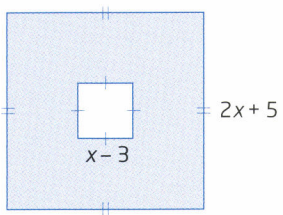
h) $100f^2 - 120fg + 36g^2$

i) $400p^3 - 900pq^2$

For help with question 7, see Example 4.

7. **a)** Find an algebraic expression for the area of the shaded region.

b) Write the area expression in factored form.



8. Determine all values of b so that each trinomial is a perfect square.

a) $y^2 + by + 121$

b) $4x^2 + bx + 25$

c) $9n^2 + bnp + 49p^2$

d) $w^2 + 10w + b$

e) $81m^2 - 90m + b$

f) $16x^2 - 88xy + b^2y^2$

9. Determine two values of k so that each trinomial can be factored as a difference of squares.

a) $m^2 - kn^2$

b) $kx^2 - 9$

c) $49c^2 - k$

10. Factor, if possible.

a) $9a^2b^2 - 24abcd + 16c^2d^2$

b) $225 - (x + 5)^2$

c) $(3c + 2)^2 - (3c - 2)^2$

d) $4x^2 + 26x + 9$

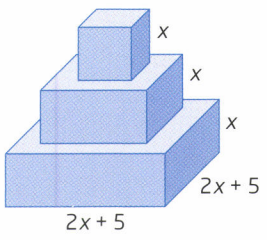
11. The area of an unknown shape is represented by $9x^2 + 30x + 25$. If x must be an integer, what shape(s) could this figure be?

12. A box is in the shape of a rectangular prism. Its volume is given as $x^3 - 2x^2 + x$.

a) Determine algebraic expressions for the dimensions.

b) Describe the faces of the box.

13. **Chapter Problem** In Section 5.3, question 12, you found an algebraic expression for the total of the top surface areas of the three prisms used to make the pedestal.



a) Write algebraic expressions for the exposed top surface areas of the middle and bottom layers of the pedestal.

b) Factor each expression from part a).

c) Compare the expressions for the exposed surface areas when x represents 5 cm.

14. The radius of a circle has been decreased by a certain amount. Its area is now given as $\pi r^2 - 14\pi r + 49\pi$, where r was the original radius, in centimetres.

a) What was the decrease in radius?

b) What was the decrease in area?

15. Is $x^2 - 1$ the same as $(x - 1)^2$? Explain using words and/or diagrams.

16. A parabola has equation $y = x^2 - 4x + 4$. Rewrite the equation in factored form to find the coordinates of the vertex.

17. Factor to evaluate each difference.

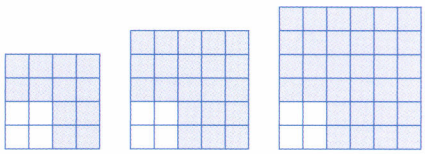
a) $15^2 - 11^2$

b) $37^2 - 27^2$

c) $98^2 - 97^2$

d) $28^2 - 22^2$

18. The first three diagrams in a pattern are shown.



a) Use a table to develop a formula to represent the number of shaded small squares in the nth diagram.

b) Write your formula in factored form.

c) Calculate the number of shaded small squares in the 10th diagram using both versions of your formula.

d) Which version is easier to use? Why?

Extend

19. A three-dimensional figure has volume given as $4\pi x^3 + 20\pi x^2 + 25\pi x$. What shape(s) could this figure be, and what are its dimensions?

20. Factor.

a) $(x - 4)^2 - 16$

b) $(x + 1)^2 + 2(x + 1) + 1$

c) $25x^4 - 9y^4$

d) $k^4 - 8k^2 + 16$

e) $a^6 + 20a^3 + 100$

f) $\frac{y^4}{81} - \frac{x^4}{625}$

21. Find all values of k so that each trinomial can be factored as a perfect square over the integers.

a) $81x^4 + kx^2 + 16$

b) $4y^4 + ky^2z^2 + 25z^4$

22. Use Technology

a) Use a CAS to factor each expression:
$x^2 - 1$
$x^3 - 1$
$x^4 - 1$
$x^5 - 1$

b) Look for a pattern in the factors. Which factored form does not appear to follow the pattern? Use a CAS to expand the last two factors of this factored form. Note what happens.

c) Use your pattern to predict the result of factoring $x^6 - 1$ into two factors. Check your prediction using a CAS. If necessary, expand factors.

23. Math Contest

a) Show that $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$.

b) Factor $m^3 - 64$.

c) Factor $27y^3 - 125z^6$.

24. Math Contest

a) Show that
$a^3 + 1000 = (a + 10)(a^2 - 10a + 100)$.

b) Factor $k^6 + 216e^3$.

c) Factor $343q^{12} + 729r^{24}$.

25. Math Contest

a) Expand $(a + b)^4$.

b) Factor.
$81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$

26. Math Contest If $a^2 + b^2 = 15$ and $ab = 3$, then the value of $(a - b)^2$ is

A 21

B 18

C 12

D 9

E 3

Chapter 5 Review

5.1 Multiply Polynomials, pages 210–219

1. Use the distributive property to find each binomial product.

a) $(x + 5)(x + 8)$ **b)** $(x - 1)(x - 4)$

c) $(x + 3y)(x - 6y)$ **d)** $(5a - 6b)(2a + 9b)$

2. Expand and simplify.

a) $-(k + 2)(k - 7)$

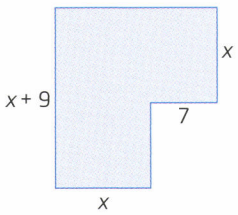
b) $w(w - 7v)(w - 3v)$

c) $7x(7x + 3y)(3x + 9y)$

d) $(y + 1)(y + 8) + (y - 8)(y + 1)$

e) $-4(2x - 6)(4x - 5) + 3(11x + 7)(11x - 4)$

3. Write an algebraic expression to represent the area of the figure. Expand and simplify.



5.2 Special Products, pages 220–227

4. Draw a diagram to illustrate each product.

a) $(x + 4)^2$ **b)** $(a + 2)^2$

5. Expand and simplify.

a) $(x + 6)^2$ **b)** $(k + 8)^2$ **c)** $(p + 7)^2$

d) $(r - 2)^2$ **e)** $(e - 9)^2$ **f)** $(q - 20)^2$

6. Expand and simplify.

a) $(b + 6)(b - 6)$ **b)** $(a - 7)(a + 7)$

c) $(y + 12)(y - 12)$ **d)** $(x - 15)(x + 15)$

e) $(e - 10)(e + 10)$ **f)** $(x + 6)(x - 6)$

7. Expand and simplify.

a) $(y + 4x)^2$

b) $(7m - n)^2$

c) $(2c + 9d)^2$

d) $-2(5x + 7y)(5x - 7y)$

e) $(3a - 8c)(3a + 8c)$

f) $-(5x - 8y)(5x + 8y)$

5.3 Common Factors, pages 228–235

8. Use algebra tiles or a diagram to illustrate the factoring of each polynomial.

a) $x^2 + 9x$ **b)** $3x^2 + x$

9. Factor.

a) $12y + 24z$ **b)** $11xy - 9xz$

c) $c^2 + 3c$ **d)** $4k - 8k^3$

10. Factor by grouping.

a) $4m^2 + 12m + 3m + 9$

b) $9k^2 - 6k + 6k - 4$

c) $8x^2 + 16x - 5x - 10$

d) $16x^2 - 12xy - 12xy + 9y^2$

11. Factor, if possible.

a) $8m^2 - 16m + 4$

b) $18c^3 + 24c^2 - 8c + 6$

c) $15m^2 - 22mn + 66mn^2$

d) $ax^2z - az^2 + axz$

12. A rectangle has area given by the expression $10x^2 + 5x$. The length and width can be found by factoring the expression. Find all possible expressions for the length and width.

5.4 Factor Quadratic Expressions of the Form $x^2 + bx + c$, pages 236–241

13. Illustrate the factoring of each trinomial using algebra tiles or a diagram.

a) $x^2 + 9x + 14$

b) $x^2 + 11x + 18$

c) $x^2 + 4x + 4$

14. Factor.

a) $c^2 - 17c + 72$ **b)** $x^2 - 8x + 15$

c) $z^2 - 14z + 33$ **d)** $x^2 + 3x - 10$

e) $x^2 + 8x - 9$ **f)** $x^2 - 2x - 8$

15. A parabola has equation $y = x^2 - 3x - 18$.

a) Factor the right side of the equation.

b) Identify the x-intercepts of the parabola.

c) Find the equation of the axis of symmetry, find the vertex, and draw the graph.

5.5 Factor Quadratic Expressions of the Form $ax^2 + bx + c$, pages 242–249

16. Factor. Use a diagram, if needed.

a) $2x^2 + 7x + 6$ **b)** $6y^2 + 29y + 9$

c) $6a^2 - 23a + 15$ **d)** $4b^2 - 5b + 1$

e) $12m^2 + 20m - 8$ **f)** $14k^2 - 31k - 10$

17. Factor, if possible.

a) $8x^2 + 2xy - 15y^2$ **b)** $5c^2 - 4cd + 9d^2$

c) $12m^2 + 13mn + 3n^2$ **d)** $2w^2 - 9wx - 8x^2$

e) $24x^2 + 6xy - 9y^2$ **f)** $9g^2 + 4g - 10$

18. A rectangle has area defined by $8x^2 + 2x - 15$.

Area is $8x^2 + 2x - 15$.

a) Factor to find algebraic expressions for the length and width of the rectangle.

b) If x represents 12 cm, determine the perimeter and the area of the rectangle.

5.6 Factor a Perfect Square Trinomial and a Difference of Squares, pages 248–255

19. Factor.

a) $x^2 - 49$ **b)** $y^2 - 121$

c) $4k^2 - 9$ **d)** $16 - 144a^2$

e) $9w^2 - 25x^2$ **f)** $1 - 81p^2$

20. Verify that each trinomial is a perfect square. Then, factor.

a) $x^2 + 14x + 49$

b) $k^2 + 8k + 16$

c) $y^2 - 10y + 25$

d) $25y^2 - 70y + 49$

e) $36c^2 - 108c + 81$

f) $121 + 176y + 64y^2$

21. Determine all values of k so that each trinomial is a perfect square.

a) $25y^2 + ky + 144$

b) $9x^2 - 42x + k$

22. Factor, if possible.

a) $9w^2x^2 - 24wxyz + 16y^2z^2$

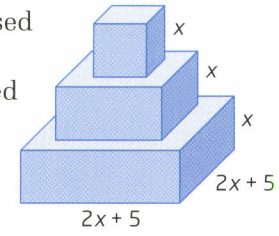
b) $625 - (y - 5)^2$

c) $144a^4 - 499b^4$

d) $25x^2 + 30x + 4$

Chapter Problem Wrap-Up

In Section 5.2, question 13, you wrote algebraic expressions for the exposed top surface areas of the middle and bottom layers of the pedestal. In Section 5.6, question 13, you factored algebraic expressions for the exposed top surface areas of the middle and bottom layers of the pedestal.



a) Consider your resulting expressions from Section 5.2. Factor these expressions.

b) Were the results the same as those from Section 5.6? Explain.

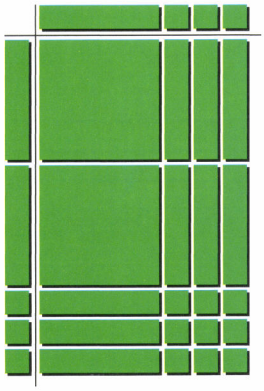
c) Calculate the total volume of wood that will be used in building the pedestal, if x represents 5 cm and if x represents 10 cm. Compare the results.

d) When you doubled the value of x in part c), did the volumes double? Why or why not?

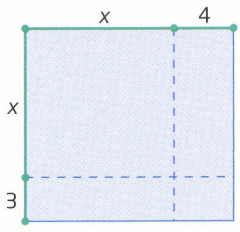
Chapter 5 Practice Test

1. What binomial product does each diagram illustrate?

a)



b)



2. Simplify.

a) $4x^2(3x - 5y + 8z)$

b) $3m(6m^2 - 5m + 4) - (4m^3 - 8m^2 + 9)$

3. Expand and simplify.

a) $(y + 5)(y + 9)$

b) $(4x - 7)(3x + 2)$

c) $(6k + 1)(6k - 1)$

d) $(w - 8)^2$

e) $(4c + 5d)^2$

f) $2(x - 4)(x - 7) - 5(8x - 9)(8x + 9)$

4. The minimum stopping distance, after a delay of 1 s, for a particular car is modelled by the formula $d = 0.006(s + 1)^2$, where d represents the stopping distance, in metres, and s represents the initial speed, in kilometres per hour.

a) Expand and simplify the formula.

b) Compare the results in both versions of the formula for an initial speed of 60 km/h.

5. Factor.

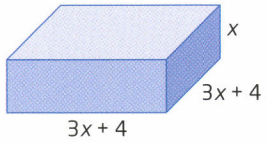
a) $9d^2e^2 + 6d^3e$

b) $15p^2qr^3 - 25p^3q^2r + 5pqr$

c) $5(x + 6) - 2(x + 6)$

d) $16x^2 + 8x - 6x - 3$

6. a) Find an algebraic expression for the surface area of the square-based prism.



b) Expand and simplify your expression from part a).

c) Factor the resulting expression from part b).

7. Factor.

a) $x^2 + 11x + 24$

b) $y^2 - 15y + 56$

c) $n^2 - n - 90$

d) $x^2 - 14x + 49$

e) $h^2 - 100$

f) $d^2 + 16d + 64$

8. Factor.

a) $3k^2 + 12km - 36m^2$

b) $8y^2 + 19y + 6$

c) $9w^2 - 24w + 7$

d) $25a^2 + 60a + 36$

e) $121w^2 - 144$

f) $10x^2 - 7xy - 6y^2$

9. Explain how to determine whether or not you can factor $9x^2 - 10x + 18$ over the integers.

10. The area of a rectangle is given as $x^2 + 13x - 30$.

a) Determine polynomials that represent the length and width of the rectangle.

b) What is the smallest integer value of x for which this area expression makes sense?

11. Determine all values of k so that each trinomial is a perfect square.

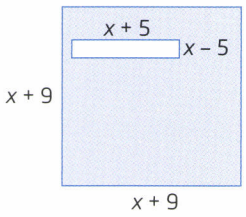
a) $36x^2 + kx + 121$

b) $49d^2 - 56d + k$

c) $25x^2 - 60xy + ky^2$

d) $ka^2 + 30ab + 9b^2$

12. a) Write an algebraic expression for the area of the shaded region.



b) Write the area expression in factored form.

c) Substitute $x = 7$ into both forms. Are the results the same? Why?

13. A parabola has equation $y = 2(x + 6)^2 - 2$.

a) Expand and simplify to write the equation in the form $y = ax^2 + bx + c$.

b) Factor your equation from part a).

c) Do the three equations represent the same parabola? Justify your response.

14. The volume of a rectangular prism is given as $9x^3 - 30x^2 + 25x$.

a) Determine algebraic expressions for the dimensions.

b) Describe the faces of the prism.

15. Determine two values of k so that each trinomial can be factored as a difference of squares.

a) $km^2 - 25$

b) $16d^2 - k$

c) $a^2 - kb^2$

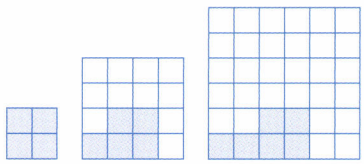
16. Factor to evaluate each difference.

a) $34^2 - 31^2$

b) $127^2 - 126^2$

c) $52^2 - 48^2$

17. The first three diagrams in a pattern are shown.



a) Write a formula for the total number of small squares in the nth diagram.

b) Write a formula for the number of shaded small squares in the nth diagram.

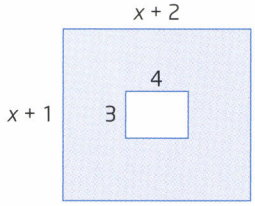
c) Write a formula for the number of unshaded small squares in the nth diagram.

d) Write your formula from part c) in factored form.

e) Show that both forms of the formula give the same results for the 15th diagram.

Achievement Check

18. a) Find all values of b so that $x^2 + bx + 10$ can be factored over the integers.

b) Find all values of b so that $4y^2 + by + 5$ can be factored over the integers.

c) Write an algebraic expression for the shaded area. Then, write the expression in factored form.